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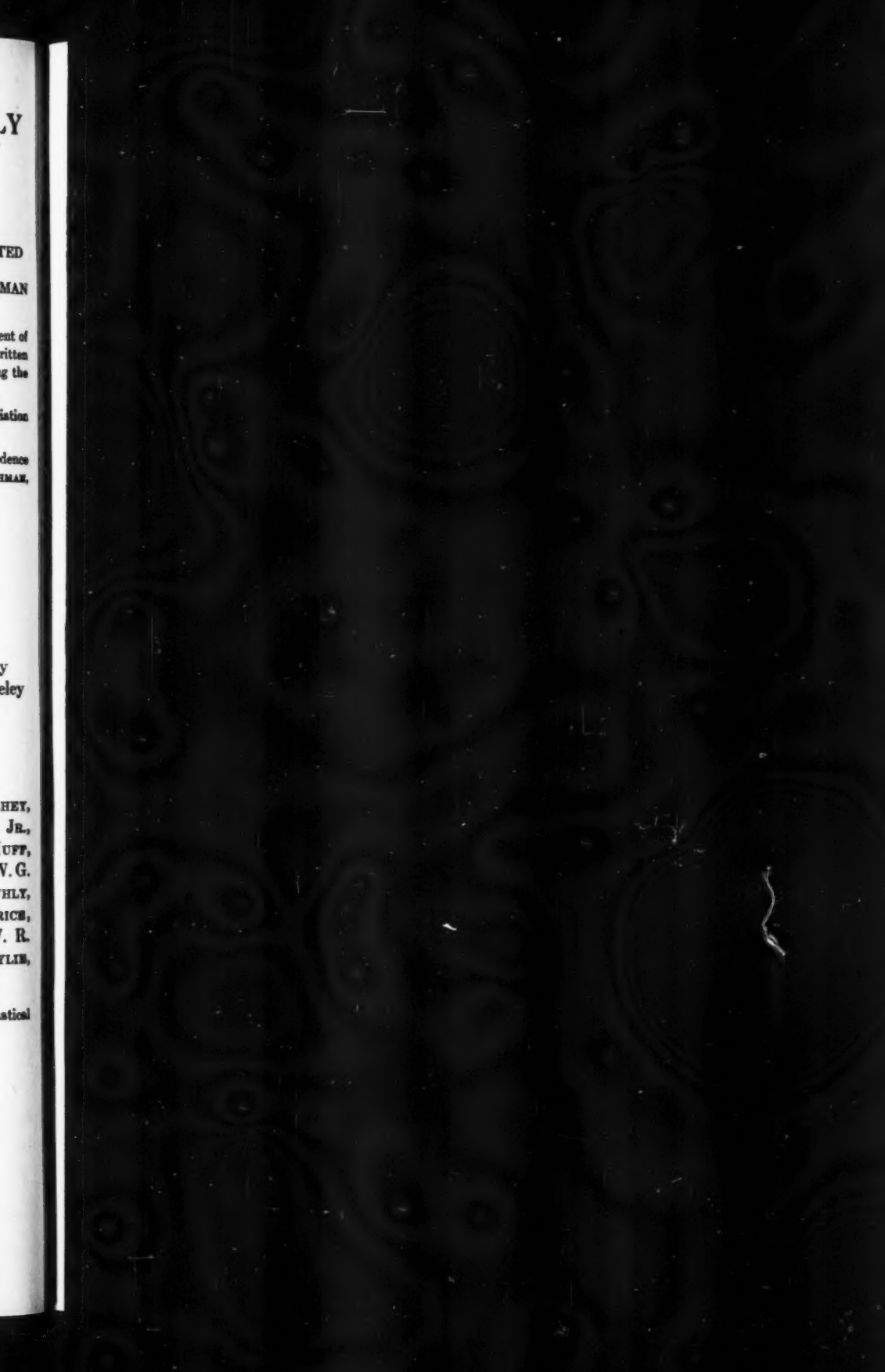

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THE CALCULATION OF THE COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

MORGAN WARD, California Institute of Technology

1. Introduction. The difficulty of computing an elliptic integral of the third kind is well known; even a computation of the complete integral

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 + \nu \sin^2 \phi)\sqrt{(1 - k^2 \sin \phi)}} \tag{1.1}$$

for numerical values of its parameters ν and k is a time-consuming operation. Apparently it is only very recently that any systematic tabulation has been made. I develop here a very efficient way to compute this integral when the parameters ν and k are real. The procedure is well adapted to a digital computer, and is so easily programmed that no tables should be necessary in the future.

When ν is zero, the procedure reduces to Gauss' method for computing the complete elliptic integral of the first kind by the use of Landen's transformation. However no knowledge of elliptic functions or integrals is required for the developments which follow. Any reader who has had a first course in function theory can easily understand the proofs.

2. The computation procedure. The process of computation is iterative, but takes slightly different forms according as the parameter ν is positive or negative. We let $\nu = \epsilon\mu^2$, $\epsilon = \pm 1$.

If $\epsilon = +1$, we speak of the positive case; if $\epsilon = -1$, of the negative case. In either case, we denote the integral (1.1) by $L(\mu, k)$. If $\mu = 0$, the integral reduces to the complete integral of the first kind. We denote it then by $L(k)$:

$$L(k) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}}. \tag{2.1}$$

The parameters μ and k are assumed to be real, and

$$0 \leqq k < 1, \qquad 0 \leqq \mu \quad \text{if} \quad \epsilon = 1; \qquad 0 \leqq \mu < 1 \quad \text{if} \quad \epsilon = -1. \tag{2.2}$$

Write for brevity

$$k' = \sqrt{(1 - k^2)}, \qquad \mu' = \sqrt{(1 + \epsilon\mu^2)}, \tag{2.3}$$

and let $k^* = (1-k')/(1+k')$ and $\mu^* = \mu k/\{(1+\mu')(1+k')\}$. We shall prove in Section 4 of this paper that

$$L(\mu, k) = \frac{2}{1 + k'} \mu'^{-1}(2L(\mu^*, k^*) - L(k^*)). \tag{2.4}$$

Now $k^* = k^2/(1+k')^2 < k^2$ so that if we iterate (2.4), the successive values of μ and k obtained tend rapidly to zero. Since $L(0, 0) = L(0) = \frac{1}{2}\pi$, the following procedure suggests itself for obtaining an approximation $A(\mu, k)$ to $L(\mu, k)$.

First step. Compute $k_1, \cdots, k_r$ *and* $\mu_1, \cdots, \mu_r$ *by the formulas*

$$(2.5)\quad \mu_1 = \mu,\ k_1 = k; \qquad k_{n+1} = \frac{1 - k_n'}{1 + k_n'}, \qquad \mu_{n+1} = \frac{\mu_n k_n}{(1 + \mu_n')(1 + k_n')}.$$

The algorithm is continued until k_{r+1} and μ_{r+1} are sufficiently small. More specifically for $n = 1, 2, \cdots$, let

$$(2.6)\qquad K_n = \prod_{s=1}^{n} 2/(1 + k_s'), \qquad M_n = \prod_{s=1}^{n} \mu_s'^{-1},$$

and let

$$(2.7)\qquad E_n = \begin{cases} 2^n K_n M_n k_{n+1}^2 & \text{in the positive case;} \\ 2^n K_n M_n (\mu_{n+1}^2 + \tfrac{1}{2} k_{n+1}^2) & \text{in the negative case.} \end{cases}$$

We require that E_r be negligible.

Second step. Compute the approximations $A(\mu_{r+1}, k_{r+1})$, $A(\mu_r, k_r)$, $A(\mu_{r-1}, k_{r-1})$, $\cdots A(\mu_1, k_1) = A(\mu, k)$ *and* $A(k_{r+1})$, $A(k_r)$, $A(k_{r-1})$, $\cdots A(k_2)$ *by the formulas*

$$A(\mu_{r+1}, k_{r+1}) = A(k_{r+1}) = \tfrac{1}{2}\pi,$$

$$(2.8)\qquad A(\mu_n, k_n) = \frac{2}{1 + k_n'} \frac{1}{\mu_n'} (2A(\mu_{n+1}, k_{n+1}) - A(k_{n+1})),$$

$$A(k_n) = \frac{2}{1 + k_n'} A(k_{n+1}), \qquad n = r, r - 1, \cdots, 1.$$

Third step. Take $A(\mu, k)$ *as an approximation to* $L(\mu, k)$; *the error in doing so is negligible, for*

$$(2.9)\qquad 0 < L(\mu, k) - A(\mu, k) < E_r.$$

It will be observed that the first two steps of the computation are iterative processes very well adapted to a digital computer or even to an ordinary desk calculator.

3. The rapidity of convergence. The convergence of the process is extremely rapid. It is evident from (2.2) and (2.5) that if μ and k are positive, then $0 < k_{n+1} < k_n^2$ and $0 < \mu_{n+1} < \frac{1}{2}(\mu_n^2 + k_n^2)$. We shall show later that *if* $0 < k_t, \mu_t < \frac{2}{3}$ *and* $n \geqq 1$, *then*

$$(3.1)\qquad 0 < k_{n+t} < (\tfrac{1}{4} k_t)^{2^n}, \qquad 0 < \mu_{n+t} < \mu_t (\tfrac{1}{4} k_t)^{2^{n-1}}.$$

We shall also show that if $0 < k_n, \mu_n < .1$, then

$$(3.2)\qquad 2/(1 + k_n') = 1 + \theta_n (\tfrac{1}{4} k_n^2), \qquad 1/\mu_n' = 1 - \epsilon\phi_n (\tfrac{1}{2} \mu_n^2),$$

with $1 < \theta_n < 1.006$, $1 < \phi_n < 1.008$.

It follows from (3.2) that the infinite products $K=\prod_1^\infty \{2/(1+k_n')\}$, $M=\prod_1^\infty (1/\mu_n')$, converge and, from (3.1), that the partial products M_r and K_r in the error term E_r change very little as r increases when μ_r and k_r are small. It is also evident from the inequalities (3.1) that E_r may be made arbitrarily small by choosing r large enough; that is, the procedure converges in the usual sense. The following numerical data for the negative case illustrate the rapidity of convergence.

The data is taken from a sytematic computation of E_r to twenty places for μ and k ranging independently over the values .1, .2, $\cdots$, .8, .9. The computation was programmed for the California Institute Datatron by Mr. George W. Logemann, and the complete results will be given elsewhere. Evidently the most unfavorable case is when $\mu=k=.9$. Mr. Logemann found that $E_3=.000174$, $\cdots$, $E_4=.00000\ 00000\ 5197$, $\cdots$, $E_5<10^{-18}$. We conclude then that four or five iterations will suffice for most ordinary purposes.

4. Derivation of the iterative formula. We here derive formula (2.4) on which the computation procedure rests.

The functions $(1+\epsilon\mu^2\sin^2\phi)^{-1}$ and $(1-k^2\sin^2\phi)^{-1/2}$ may evidently be expanded in Fourier series which converge uniformly to them in the closed interval $[-\pi,\pi]$. Let these expansions be

$$\frac{1}{1+\epsilon\mu^2\sin^2\phi}=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos n\phi, \tag{4.1}$$

$$\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_n\cos n\phi. \tag{4.2}$$

Then

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos n\phi d\phi}{\sqrt{(1-k^2\sin^2\phi)}}. \tag{4.3}$$

It will be shown at the close of this section that $a_{2n+1}=0$ and that

$$a_{2n}=2\mu'^{-1}\epsilon^n\cdot\{\mu/(1+\mu')\}^{2n}. \tag{4.4}$$

Thus by (4.1)

$$\frac{1}{(1+\epsilon\mu^2\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}}$$

$$=\mu'^{-1}\left(\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}+2\sum_1^{\infty}\epsilon^n\left(\frac{\mu}{1+\mu'}\right)^{2n}\frac{\cos 2n\phi}{\sqrt{(1-k^2\sin^2\phi)}}\right).$$

The series on the right is uniformly convergent with respect to ϕ. Hence integrating term-wise over $[-\pi,\pi]$ we obtain from (2.1) and (4.3) the formula

$$L(\mu, k) = \tfrac{1}{4}\pi\mu'^{-1}\left(b_0 + 2\sum_1^\infty \epsilon^n\{\mu/(1+\mu')\}^{2n}b_{2n}\right). \tag{4.5}$$

We next obtain another expression for the coefficients b_{2n} in this series. Let $w=\phi+i\zeta$ be a complex variable. Then there exists a positive number δ depending on k alone such that $\Delta(w)=(1-k^2\sin^2 w)^{-1/2}$ is regular in the strip $|\operatorname{Im} w|<\delta$. Since $\Delta(w)$ is also periodic with period 2π in the whole plane, it is representable everywhere in the strip by the Fourier series

$$\Delta(w) = \tfrac{1}{2}b_0 + \sum_1^\infty b_n \cos nw, \tag{4.6}$$

with the b_n given by formula (4.3). From now on, we confine ourselves to the rectangle $|\operatorname{Re} w|\leqq\pi$, $|\operatorname{Im} w|<\delta$.

Let T stand for the mapping of the w-plane induced by letting $z=e^{iw}$. Then T transforms the rectangle above into a circular ring R in the z plane: $R=\{z\,|\,0<r<|z|<R\}$, where $e^{-\delta}=r<1<R=e^{\delta}$. Furthermore if

$$F(z) = T\Delta(w), \tag{4.7}$$

then $F(z)$ is regular in R. But $T\cos nw=\frac{1}{2}(z^n+z^{-n})$. Hence if we let $b_{-n}=b_n$, we have by (4.6) and (4.7) the Laurent expansion of $F(z)$ in R: $F(z)=\sum_{-\infty}^{\infty}\frac{1}{2}b_nz^n$.

Consequently by Laurent's theorem,

$$b_n = \frac{1}{\pi i}\int_\Gamma z^{n-1}F(z)dz, \qquad n = 0, 1, \cdots. \tag{4.8}$$

Here Γ is the unit circle described once counterclockwise. Now let

$$k = \sin\tfrac{1}{2}\beta, \qquad 0 \leqq \beta < \pi. \tag{4.9}$$

Then we find from (4.7) after a little algebra that

$$F(z) = \frac{2\csc\frac{1}{2}\beta z}{\sqrt{\{(z^2+\tan^2\frac{1}{4}\beta)(z^2+\cot^2\frac{1}{4}\beta)\}}}. \tag{4.10}$$

Hence the only singularities of $F(z)$ within Γ are branchpoints of order two at $z=\pm i\tan\frac{1}{4}\beta$. Join these points by a cut along the axis of imaginaries. Then the contour Γ in (4.8) may be shrunk down to a path consisting of two small circles about the branch points and the two sides of the cut. If the radius of each of these circles is ρ, their contribution to the line integral is of order $\sqrt{\rho}$. If therefore we set $z=i\tan\frac{1}{4}\beta\sin\phi$ along the cut and pass to the limit by letting ρ tend to zero, we find that

$$b_n = \frac{4}{\pi}i^n\csc\tfrac{1}{2}\beta\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin^n\phi\,\tan^{n+1}\frac{1}{4}\beta d\phi}{\sqrt{(1-\tan^4\frac{1}{4}\beta\sin^2\phi)}}.$$

Hence b_n is zero when n is odd, and when n is even,

$$b_{2n} = \frac{8}{\pi}\,\frac{(-k^*)^n}{1+k'}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi\,d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}. \tag{4.11}$$

Here we have written k^* for $\tan^2\frac{1}{4}\beta$ and replaced the multiplier $\csc\frac{1}{2}\beta\tan\frac{1}{4}\beta$ by $(1+\cos\frac{1}{2}\beta)^{-1}$ which equals $(1+k')^{-1}$ by (4.9) and (2.3). We easily find from the same formulas that

$$k^* = (1-k')/(1+k'). \tag{4.12}$$

(4.11) is the new expression for b_{2n} which we were seeking. Introduce it into the series (4.5) for $L(\mu, k)$, and let

$$\mu^* = \frac{\mu}{1+\mu'}\sqrt{k^*} = \frac{\mu}{1+\mu'}\,\frac{k}{1+k'}. \tag{4.13}$$

Then (4.5) becomes

$$L(\mu,k) = \frac{2\mu'^{-1}}{1+k'}\left(\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}} + 2\sum_1^\infty(-\epsilon)^n\mu^{*2n}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}\right). \tag{4.14}$$

Now the series $(1-k^*\sin^2\phi)^{-1/2}+2\sum_1^\infty(-\epsilon\mu^{*2})^n\sin^{2n}\phi(1-k^*\sin^2\phi)^{-1/2}$ is uniformly convergent in ϕ, and is essentially a geometric progression whose sum may be written

$$\frac{2}{(1+\epsilon\mu^{*2}\sin^2\phi)\sqrt{(1-k^{*2}\sin^2\phi)}} - \frac{1}{\sqrt{(1-k^{*2}\sin^2\phi)}}.$$

Hence on integrating term-wise from 0 to $\frac{1}{2}\pi$ and comparing it with (4.14) and the integral of its sum with (1.1) and (2.1), we obtain the iteration formula (2.4), with the values of k^* and μ^* given in Section 2.

It remains to verify the values given for the Fourier coefficients a_n in the series (3.1) for $(1+\epsilon\mu^2\sin^2\phi)^{-1}$. Let $\mu=\tan\frac{1}{4}\alpha$, $0\leqq\alpha<\pi$, in the positive case; $\mu=\sin\frac{1}{2}\alpha$, $0\leqq\alpha<\pi$, in the negative case. Then we find that

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \begin{cases} 4\cot^2\frac{1}{2}\alpha z^2[(z^2-\tan^2\frac{1}{4}\alpha)(z^2-\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=+1,\\ 4\csc^2\frac{1}{2}\alpha z^2[(z^2+\tan^2\frac{1}{4}\alpha)(z^2+\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=-1.\end{cases}$$

This result may be written

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \frac{4\mu^{-2}z^2}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}.$$

Hence by analogy with (4.8)

$$a_n = \frac{4\mu^{-2}}{\pi i}\int_\Gamma\frac{z^{n+1}dz}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}.$$

Thus by the residue theorem a_n equals $8\mu^{-2}$ times the sum of the residues of the integrand at its two poles $\pm\sqrt{\epsilon}\tan\frac{1}{4}\alpha$ within Γ. On evaluating these residues, we find that $a_{2n+1}=0$ and that a_{2n} has the value stated in (4.4), so that the proof is complete.

5. Proofs of the basic inequalities. The inequalities (3.1) are proved as follows. We have $k_{t+1}=(1-k_t')/(1+k_t')=k_t^2/(1+k_t')^2$. Hence $k_{t+1}<\frac{1}{2}k_t^2$ if and only if $1+k_t'>\sqrt{3}$. This works out to $k_t^2<\sqrt{(3+2\sqrt{3})}=.681$. Hence: *If* $k_t<\frac{2}{3}$, *then* $k_{t+1}<\frac{1}{2}k_t^2$. This statement is the case $n=1$ of the inequality (3.1) for k_{t+n}. The general inequality follows by an easy induction. The inequality for μ_{t+n} is proved similarly.

To prove that (3.2) is valid, assume that $0<k_n\leq 1$. Then

$$\frac{2}{1+k_n'}=\frac{2(1-k_n')}{1-k_n'^2}=\frac{2}{k_n^2}\sqrt{(1-k_n^2)}).$$

Now on expanding the function $1-\sqrt{(1-x)}$, $0<x<1$ by Taylor's theorem with remainder and then substituting k_n^2 for x, we obtain the formulas

$$\frac{2}{1+k_n'}=\theta_n(\tfrac{1}{4}k_n),\quad \theta_n=1+\tfrac{1}{2}k_n^2+\frac{5}{16}\frac{k_n^4}{(1-\theta k_n^2)^{7/2}},\qquad 0<\theta<1.$$

Hence

$$1<\theta_n<1+\tfrac{1}{2}(.01)+\frac{5}{16}\frac{.0001}{(.99)^{7/2}}<1.006.$$

The formula for μ_n' in (3.2) is proved in a similar fashion.

We preface the proof of the inequality (2.9) by two lemmas.

LEMMA 5.1. *If* $0<\mu,\ k<.1$, *then*

(5.1) $\qquad 0<L(\mu,k)-\frac{1}{2}\pi<\frac{4}{5}k^2$ *if* ϵ *is positive,*

(5.2) $\qquad 0<L(\mu,k)-\frac{1}{2}\pi<\frac{4}{5}(\mu^2+\frac{1}{2}k^2)$ *if* ϵ *is negative.*

Proof. By Taylor's theorem with remainder, where $0<\theta<1$,

$$(1-k^2\sin^2\phi)^{-1/2}=1+\tfrac{1}{2}k^2\sin^2\phi+\frac{3}{8}k^4\frac{\sin^4\phi}{(1-\theta k^2\sin^2\phi)^{5/2}}$$

$$>1+\tfrac{1}{2}k^2\left(1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}\right)\sin^2\phi.$$

Since $k<.1$,

$$1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}<1+\frac{3}{4}\frac{.01}{(.99)^{5/2}}=1.0077.$$

Hence

$$\frac{1}{(1-k^2\sin^2\phi)^{1/2}} < 1+\alpha(\tfrac{1}{2}k^2)\sin^2\phi, \qquad \alpha = 1.0077. \tag{5.3}$$

Similarly

$$\frac{1}{1-\mu^2\sin^2\phi} < 1+\beta\mu^2\sin^2\phi, \qquad \beta = 1.01031.$$

Therefore

$$0 < \frac{1}{(1-\mu^2\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}} - 1$$
$$< \tfrac{1}{2}\alpha k^2\sin^2\phi + \beta\mu^2\sin\phi + \alpha\beta(\tfrac{1}{2}\mu^2k^2)\sin^4\phi.$$

On integrating these inequalities from 0 to $\frac{1}{2}\pi$ we obtain when ϵ is negative the inequality

$$0 < L(\mu, k) - \tfrac{1}{2}\pi < \tfrac{1}{4}\pi\alpha(1+\tfrac{3}{8}\beta\mu^2)(\tfrac{1}{2}k^2) + \tfrac{1}{4}\pi(1+\tfrac{3}{16}k^2\alpha)\mu^2.$$

The coefficients of $\frac{1}{2}k^2$ and μ^2 turn out to be .79446 and .794996 which are both less than $\frac{4}{5}$. This completes the proof of the inequality (5.2).

If ϵ is positive, it follows from (5.3) that

$$L(\mu, k) - \tfrac{1}{2}\pi < L(k) - \tfrac{1}{2}\pi - (\tfrac{1}{2}\alpha k^2)(\tfrac{1}{4}\pi).$$

Since $\frac{1}{4}\alpha\pi < \frac{4}{5}$, (5.1) follows, completing the proof of the lemma.

LEMMA 5.2. *If*

$$A(\mu_n, k_n) = \frac{2\mu_n'^{-1}}{1+k_n'}\,(2a(\mu_{n+1}, k_{n+1}) - A(k_{n+1})),$$

$$A(k_n) = \frac{2}{1+k_n'}\,A(k_{n+1}),$$

for $n = 1, 2, \cdots$, *then*

$$L(\mu, k) - A(\mu, k) = 2^nM_nK_n(L(\mu_{n+1}, k_{n+1}) - A(\mu_{n+1}, k_{n+1}))$$
$$- \sum_{s=1}^{n} 2^{s-1}M_sK_s(L(k_{s+1}) - A(k_{s+1}), \tag{5.2}$$

where K_n *and* M_n *are given for all* n *by* (2.6).

Proof. We know by the iteration formula that for all n,

$$L(\mu_n, k_n) = \frac{2}{1+k_n'}\,\mu_n'^{-1}(2L(\mu_{n+1}, k_{n+1}) - L(k_{n+1}),$$

$$L(k_n) = \frac{2}{1+k_n'} L(k_{n+1}).$$

Hence since $\mu = \mu_1$ and $k = k_1$,

$$\begin{aligned} L(\mu, k) - A(\mu, k) &= \frac{2}{1-k_n'} \mu_1'^{-1}(2L(\mu_2, k_2) - L(k_2) - \frac{2}{1+k_1'} \mu_1'^{-1}(2A(\mu_2, k_2) - A(k_2)) \\ &= 2K_1M_1(L(\mu_2 k_2) - A(\mu_2, k_2) - K_1M_1(L(k_2) - A(k_2)). \end{aligned}$$

Hence (5.2) is true when $n=1$. But is easy to show that if it is true for n, it is true for $n+1$. Hence it is true for all n by mathematical induction. Now

$$L(k_{s+1}) = \frac{2}{1+k_{s+1}'} \frac{2}{1+k_{s+2}'} \cdots \frac{2}{1+k_n'} L(k_{n+1}),$$

$$A(k_{s+1}) = \frac{2}{1+k_{s+1}'} \frac{2}{1+k_{s+2}'} \cdots \frac{2}{1+k_n'} A(k_{n+1}).$$

Hence (5.2) may be written

$$\begin{aligned} L(\mu, k) - A(\mu, k) &= 2^n K_n M_n (L(\mu_{n+1} k_{n+1}) - A(\mu_{n+1}, k_{n+1}) \\ &\quad - K_n \Big(\sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(k_{n+1}) - A(k_{n+1}). \end{aligned} \tag{5.3}$$

Now suppose that $A(k_{n+1}, \mu_{n+1}) = A(k_{n+1}) = \frac{1}{2}\pi$. Then if ϵ is positive, $L(\mu_{n+1}, k_{n+1}) < L(k_{n+1})$. Hence when ϵ is positive, we obtain from (5.3) the inequalities.

$$\begin{aligned} L(\mu, k) - A(\mu, k) &< 2^n K_n M_n (L(k_{n+1}) - \tfrac{1}{2}\pi) \\ &> K_n \Big(2^n M_n - \sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(K_{n+1}, \mu_{n+1}) - \tfrac{1}{2}\pi). \end{aligned}$$

Now $M_1 < M_2 < \cdots < M_n$. Hence $\sum_{s=1}^{n} 2^{s-1} M_s < 2^n M_n$. Hence $L(\mu, k) - A(\mu, k)$ is positive from the second inequality. And by the first inequality and Lemma 5.1, it is less than $2^n K_n (\frac{2}{3} k_{n+1}^2)$.

If ϵ is negative, $L(\mu_{n+1}, k_{n+1}) > L(k_{n+1})$. Hence we obtain from (5.3) the inequalities

$$\begin{aligned} L(\mu, k) - A(\mu, k) &> K_n \Big(2^n M_n - \sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(k_{n+1}) - \tfrac{1}{2}\pi) > 0, \\ L(\mu, k) - A(\mu, k) &< 2^n K_n M_n (L(\mu_{n+1}, k_{n+1}) - \tfrac{1}{2}\pi) \\ &< 2^n K_n M_n \tfrac{4}{3} (\mu_{n+1}^2 + \tfrac{1}{2} k_{n+s}^2), \end{aligned}$$

by Lemma 5.1, and this completes the proof of the inequality (2.9).

In conclusion, it is worth noting that the method may evidently be extended

to compute a complete integral of the third kind of the form

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1+\nu\sin^2\phi)\sqrt{(1+k^2\sin^2\phi)}}$$

for ν and k real and ν greater than -1.

ANALYTICAL EXPRESSIONS AND ELEMENTARY FUNCTIONS

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The title is very nearly meaningless. When Professor Ritt discusses integration in finite terms an elementary function is one (a not too elementary) thing. In common classroom usage it is another—roughly, any differentiable function frequently used by Euler. When an author presents us with an analytic expression for a function, the expression usually has little or nothing to do with analytic functions in any strict sense. The adjective has become a synonym for "nonverbal". As examples we give*

$$\operatorname{sgm}\sin^2\frac{(n-1)!+1}{n}\pi \quad \text{and} \quad \operatorname{sgm}\sin^2\frac{(n-1)!^2}{n}\pi,$$

characteristic functions for the sets of nonprime and prime integers. The spirit of the formula constructing game is clear. Its rules are not.

Consider $[x]$,† the characteristic of x regarded as a logarithm, the greatest integer not greater than x. We find

$$[x] = x - \tfrac{1}{2} + (1/\pi)\tan^{-1}\cot\pi x \qquad x \neq 0, \pm 1, \pm 2, \cdots.$$

How do we fill in the gaps at integral x? Using limits is too much like hunting quail with an elephant gun. It's difficult to find functions which are not expressible as limits of limits.

The admission of one simple additional function somehow seems more sporting. Logical candidates are null $x=\lim_{n\to\infty}(1+|x|)^{-n}$, the characteristic function of the set (0), or $\operatorname{sgm} x=\lim_{n\to\infty} x^{1/(2n+1)}=x/(|x|+\text{null } x)$. Thus, for $T(x)=(2/\pi)\sin^{-1}\sin\frac{1}{2}\pi x \operatorname{sgm}\cos\frac{1}{2}\pi x$, we have $[x]=x-\max(T(x), T(x+1))$. Even this procedure may be circumvented if we adopt the convention that an expression meaningless for $x=a$, but suitably defined in a neighborhood, defines $f(a)=\frac{1}{2}\{f(a-)+f(a+)\}$. Then $\operatorname{sgm} x=x/|x|$ and $T(x)=(2/\pi)\tan^{-1}\tan\frac{1}{2}\pi x$.

It is interesting to see how far one may go with modest assumptions. If we agree $\cos x$, $\cos^{-1} x$, x, and real constants are "common" functions of x and

* Cf. Louis Brand, Advanced Calculus, New York, 1955, p. 84.

† Cf. Arthur Porges, An analytical expression for $[X]$, this MONTHLY, vol. 66, 1959, pp. 706–707.

postulate that sums, differences, products, quotients, and common functions of common functions are common, we obtain the following:

1. $\sin x = \cos(\frac{1}{2}\pi - x)$ is common.
2. The trigonometric functions are common.
3. $\sin^{-1} x = \frac{1}{2}\pi - \cos^{-1} x$ is common.
4. $\sqrt{x} = \cos(\frac{1}{2}\cos^{-1}(2x-1))$, $0 \leqq x \leqq 1$, is common.
5. $\sqrt{x} = \sqrt{(x/x+1)}/\sqrt{(1/x+1)}$, $0 \leqq x < \infty$, is common.
6. For positive x, integral p, and q a power of 2, $x^{p/q}$ is common.
7. $\tan^{-1} x = \sin^{-1}\{x/\sqrt{(1+x^2)}\}$ is common.
8. The inverse trigonometric functions are common.
9. $|x| = \tan \cos^{-1} \cos \tan^{-1} x = \sqrt{x^2}$ is common.
10. For common $f(x)$ and $g(x)$, $2 \max(f(x), g(x)) = f(x) + g(x) + |f(x) - g(x)|$ is common. Maxima and minima of sets of functions are common.
11. The Heaviside function (not defined for $x = 0$), $h(x) = (x + |x|)/(2|x|)$, is common.
12. Characteristic functions of intervals, such as $g(x) = \min(h(x-a), h(-x+b))$, are common.
13. Piecewise common functions are common. For example, if $f(x) = f_i(x)$ on the interval $a_{i-1} < x < a_i$ which has $g_i(x)$ as characteristic function, then $f(x) = \sum_{i=1}^{n} f_i(x) g_i(x)$.

Although this last function is not defined at points a_i, removable discontinuities may be removed with a little more effort.* First, a reflection type procedure allows us to extend the domain of any common function to the entire x-axis. Thus $f(x)$ defined on $a \leqq x \leqq \infty$ coincides with $s(x) = f(|x-a| + a)$ on that interval and the latter is defined for all x. Similarly, we may extend $f(x)$ defined on $a \leqq x \leqq b$ to the function $s(x) = f(\phi(x))$, where

$$\phi(x) = \frac{a+b}{2} + \frac{b-a}{\pi} \sin^{-1} \sin\left(\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right).$$

Now we affix to the graph of $f(x)$ steeply slanting half-lines. Choose m large, let $r(x) = f(a) + m(x-a)$, $t(x) = f(b) - m(x-b)$, and define $f^*(x) = \min(r(x), s(x), t(x))$. If $f(x)$ is common on $a \leqq x \leqq b$, and $g(x)$ is common on $b \leqq x \leqq c$, and $f(b) = g(b)$, then $\max(f^*(x), g^*(x))$ is common on $a \leqq x \leqq c$, is defined for $x = b$, and coincides with $f(x)$ or $g(x)$ to the left or right of $x = b$.

If it is desired that piecewise common functions be defined at the points where jump discontinuities occur, it is necessary to accept, say, sgm x as common. Then null $x = 1 + \min(\text{sgm } x, \text{sgm } (-x))$ is common. The Heaviside function $H(x) = \frac{1}{2}(1 + \text{sgm } x - \text{null } x)$ is common, as well as characteristic functions of open intervals, $G(x) = \min(H(x-a), H(-x+b))$. The piecewise common functions are common since they may be written as $\sum f_i(x) G_i(x) + \sum k_i \text{ null } (x - a_i)$. By means of the latter term, these are given arbitrary values k_i at the juncture points a_i.

* An exception occurs when $f'(a_i+)$ and $f'(a_i-)$ are both ∞ or both $-\infty$.

SOME PROPERTIES AND APPLICATIONS OF DOUBLY STOCHASTIC MATRICES*

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1. Introduction. Since the appearance of a paper by H. Weyl [30] in 1949 there has been a considerable amount of effort, on the part of many authors, devoted to questions concerning the relationships among eigenvalues, singular values, and quadratic forms associated with a linear transformation. Almost all of these results can be formulated in terms of a single general extreme value problem involving doubly stochastic matrices. In what follows, we will indicate how this has been done and we will also discuss the current state of a pair of conjectures of van der Waerden and Kakutani on doubly stochastic matrices.

An n-square matrix $S=(s_{ij})$ is called *doubly stochastic* (d.s.) if

$$s_{ij} \geqq 0, \qquad \sum_{j=1}^{n} s_{ij} = 1, \qquad \sum_{i=1}^{n} s_{ij} = 1, \qquad 1 \leqq i, j \leqq n. \tag{1.1}$$

The totality of n-square d.s. matrices will be denoted by Ω_n. The conditions (1.1) immediately imply that Ω_n is convex and it is readily checked that if P is a permutation matrix then $P \in \Omega_n$. As we shall see, Ω_n is actually a polyhedron with the permutation matrices as vertices. (Birkhoff's theorem).

2. The Birkhoff theorem. The result of Birkhoff [3] goes as follows:

Any doubly stochastic n-square matrix S is in the convex hull of m permutation matrices for $m \leqq (n-1)^2+1$.

Several proofs and generalizations of this result are known [4], [22], [23]. They all depend on showing that there is a permutation σ of $1, \cdots, n$ such that $s_{i\sigma(i)}>0$ for $i=1, \cdots, n$. For if this can be shown let $t=\min_i s_{i\sigma(i)}>0$ and let P be a permutation matrix with $P_{i\sigma(i)}=1$, $i=1, \cdots, n$. Then if $t<1$ (if $t=1$, $S=P$ follows from (1.1)) the matrix $(1-t)^{-1}(S-tP)$ has at least one more zero entry than S. The proof is completed then by an induction on the number of positive entries of S. Mirsky showed directly that a σ with the desired properties exists and the result also follows immediately from the Frobenius-König theorem as in [4]. The condition (1.1) shows that Ω_n is of dimension $(n-1)^2$ and thus in general $m \leqq (n-1)^2+1$.

The way in which we shall use the result is as follows: suppose f is a convex (concave) function on Ω_n to the reals; then f will assume its maximum (minimum) value at a vertex of Ω_n; *i.e.*, $f(S) \leqq f(P)$ $(f(S) \geqq f(P))$ for any $S \in \Omega_n$ and P an appropriate permutation matrix.

3. Quadratic forms and eigenvalues. Let A be any normal transformation on unitary n-space to itself and let $u_1, \cdots, u_n$ be an orthonormal (o.n.) set of

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eigenvectors of A corresponding respectively to the eigenvalues $\lambda_1, \cdots, \lambda_n$. If $x_1, \cdots, x_n$ is any other o.n. set then $S=(|(x_i, u_j)|^2)$, $i, j=1, \cdots, n$, is in Ω_n. Moreover, if $z=((Ax_1, x_2), \cdots, (Ax_n, x_n))$ and $\lambda=(\lambda_1, \cdots, \lambda_n)$ then

$$z = S\lambda. \tag{3.1}$$

The relation (3.1) can be exploited immediately to yield an interesting extremal result due to K. Fan [5]:

If A is positive semidefinite Hermitian (p.s.d.) *with eigenvalues* $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n$ *then*

$$\prod_{j=1}^{k} (Ax_j, x_j) \geqq \prod_{j=1}^{k} \lambda_j, \qquad 1 \leqq k \leqq n, \tag{3.2}$$

for any o.n. *set* $x_1, \cdots, x_k$.

To see this simply set $g(z)=g(z_1, \cdots, z_n)=(\prod_{j=1}^{k} z_j)^{1/k}$ and observe that $g(z)$ is a concave function for nonnegative variables z_j. Also $f(S)=g(S\lambda)$ is concave in S for $S \in \Omega_n$. It follows that

$$\left(\prod_{j=1}^{k} (Ax_j, x_j) \right)^{1/k} = f(S) \geqq f(P) \geqq \left(\prod_{j=1}^{k} \lambda_j \right)^{1/k},$$

where P is a permutation matrix in Ω_n. From (3.2) we have

$$\prod_{j=1}^{n} a_{jj} \geqq \det A \quad (\textit{Hadamard determinant theorem})$$

by setting $k=n$ and $x_i=\epsilon_i$, the unit vector with 1 in position i. Also, if B is a second p.s.d. matrix then

$$\begin{aligned} (\det (A + B))^{1/n} &= \left(\prod_{j=1}^{n} [(Ax_j, x_j) + (Bx_j, x_j)] \right)^{1/n} \\ &\geqq \left(\prod_{j=1}^{n} (Ax_j, x_j) \right)^{1/n} + \left(\prod_{j=1}^{n} (Bx_j, x_j) \right)^{1/n} \\ &\geqq (\det A)^{1/n} + (\det B)^{1/n} \quad (\textit{Minkowski determinant inequality}), \end{aligned}$$

where $x_1, \cdots, x_n$ is an o.n. set of eigenvectors of $A+B$.

The result (3.2) has been extended by several authors in two directions: first to more complicated functions of the quadratic forms and second to a wider class of matrices than the p.s.d. ones [24], [12], [14], [15], [16], [17], [6], [7], [8].

4. Sets associated with Ω_n and the result of H. Weyl. Let $b_1 \geqq \cdots \geqq b_n$ be an ordered set of real numbers and define $\Omega_n(b)$ to be the set of all real points $t=(t_1, \cdots, t_n)$ satisfying

$$\sum_{j=1}^{k} t_{i_j} \leqq \sum_{j=1}^{k} b_j, \qquad \sum_{j=1}^{n} t_j = \sum_{j=1}^{n} b_j, \tag{4.1}$$

for all index sets $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq k \leqq n-1$. It is simple to verify that $\Omega_n(b)$ is a compact convex set. However a much stronger result of Hardy, Littlewood and Pólya [9] is the following:

$\Omega_n(b)$ is the set of all vectors Sb for $S \in \Omega_n$.

Several proofs of this are available in [9], [24], [23], [2], [10]. A reasonably succinct argument can be made in terms of support functions and by induction on n. For, denote by $L_n(b)$ the convex hull of the points Pb for P a permutation matrix. The support function of $L_n(b)$ is then $f(u) = \max_P (u, Pb)$ and thus $L_n(b)$ is the intersection of the half spaces $(u, t) \leqq f(u)$. By setting $u_0 = \epsilon_{i_1} + \cdots + \epsilon_{i_k}$ (again ϵ_{i_j} is the unit vector with 1 in position i_j and 0 elsewhere), where $i_1, \cdots, i_k$ is an index set in (4.1), we can check that if $t \in L_n(b)$ then

$$\sum_{j=1}^{k} t_{i_j} = (u_0, t) \leqq f(u_0) = \max_P (u_0, Pb) = \sum_{j=1}^{k} b_j,$$

with equality for $k = n$. Hence $L_n(b) \subseteq \Omega_n(b)$. Now suppose $t \in \Omega_n(b)$ and set $g(t) = (u, t)$ for a fixed u. Now $g(t)$ is a linear function of t defined on the set of t given by (4.1). Therefore it assumes its maximum value for $t = \bar{t}$ on one of the bounding hyperplanes in (4.1), say

$$\sum_{j=1}^{k} t_{i_j} = \sum_{j=1}^{k} b_j, \qquad 1 \leqq k \leqq n-1.$$

Let $\bar{t}^1 = (\bar{t}_{i_1}, \cdots, \bar{t}_{i_k})$, $\bar{t}^2 = (\bar{t}_{i_{k+1}}, \cdots, \bar{t}_{i_n})$, $b^1 = (b_1, \cdots, b_k)$, $b^2 = (b_{k+1}, \cdots, b_n)$ where $i_{k+1} < \cdots < i_n$ is the set of integers in $1, \cdots, n$ complementary to $i_1, \cdots, i_k$. We check that $\bar{t}^1 \in L_k(b^1)$ and $\bar{t}^2 \in L_{n-k}(b^2)$. For suppose $\alpha_1, \cdots, \alpha_r$ is a subset of the integers $i_1, \cdots, i_k$. Then $\sum_{j=1}^{r} \bar{t}_{\alpha_j} \leqq \sum_{j=1}^{r} b_j$ and also if $\beta_1, \cdots, \beta_\rho$ is a subset of $i_{k+1}, \cdots, i_n$ then

$$\sum_{j=1}^{\rho} \bar{t}_{\beta_j} + \sum_{j=1}^{k} \bar{t}_{i_j} \leqq \sum_{j=1}^{\rho+k} b_j.$$

Hence

$$\sum_{j=1}^{\rho} \bar{t}_{\beta_j} \leqq \sum_{j=1}^{\rho+k} b_j - \sum_{j=1}^{k} \bar{t}_{i_j} = \sum_{j=1}^{\rho+k} b_j - \sum_{j=1}^{k} b_j = \sum_{j=k+1}^{\rho+k} b_j.$$

Since $k < n$ and $n-k < n$ we may use induction to conclude that $\bar{t}^1 = S_1 b^1$ and $\bar{t}^2 \in S_2 b^2$ for $S_1 \in \Omega_k$, $S_2 \in \Omega_{n-k}$. Then since $i_1, \cdots, i_n$ is simply a permutation of $1, \cdots, n$ we have that $\bar{t} = Q(S_1 b^1 \dotplus S_2 b^2) = Q(S_1 \dotplus S_2) b \in L_n(b)$ for Q an appropriate permutation matrix ($\dotplus$ indicates direct sum). But then $(u, t) = g(t) \leqq g(\bar{t}) = (u, \bar{t}) \leqq f(u)$; so $(u, t) \leqq f(u)$ for all u and $t \in L_n(b)$.

A. Horn obtained the considerably stronger result that

If $t \in \Omega_n(b)$ then $t = Sb$ where S is an orthostochastic (o.s.) *matrix.*

An o.s. matrix S is one for which there exists an orthogonal matrix U such that $s_{ij} = u_{ij}^2$. The following example due to A. J. Hoffman shows that not every d.s. matrix is o.s.:

$$S = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Horn uses his result to obtain the interesting fact that the *set of all main diagonals of n-square rotation matrices is the convex hull of the points* $(\pm 1, \cdots, \pm 1)$ *of which an even number of coordinates are* -1. Also in this connection some information concerning the structure of points lying on the support planes of $\Omega_n(b)$ is known [13].

Now let A be an arbitrary nonsingular n-square matrix with eigenvalues λ_j ordered according to absolute value: $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$. Let $\alpha_1 \geqq \cdots \geqq \alpha_n$ be the positive square roots of the eigenvalues of AA^* (the α_j are called the *singular values* of A). In 1949 H. Weyl [30] proved that

$$\prod_{j=1}^{k} |\lambda_j| \leqq \prod_{j=1}^{k} \alpha_j, \qquad 1 \leqq k \leqq n \tag{4.2}$$

with equality for $k = n$. Weyl established (4.2) by comparing the dominant eigenvalue and singular value of the kth compound of A. By taking logarithms on both sides of (4.2) and letting $\log |\lambda| = (\log |\lambda_1|, \cdots, \log |\lambda_n|)$, $\log \alpha = (\log \alpha_1, \cdots, \log \alpha_n)$ we have from the Hardy, Littlewood, Pólya result that

$$\log |\lambda| = S \log \alpha \tag{4.3}$$

for $S \in \Omega_n$. As an application of (4.3) we have a special case of a result of Fan [7] proved also by Pólya [25] and originally by Weyl [30] under additional hypotheses:

$$\sum_{j=1}^{k} |\lambda_j|^s \leqq \sum_{j=1}^{k} \alpha_j^s, \qquad 1 \leqq k \leqq n,\ s > 0. \tag{4.4}$$

The result in (4.4) follows easily by defining the convex function $g(t) = \sum_{j=1}^{k} e^{t_j}$ and noting that

$$\sum_{j=1}^{k} |\lambda_j|^s = g(\log |\lambda|) = g(S \log \alpha) \leqq g(P \log \alpha) \leqq \sum_{j=1}^{k} \alpha_j^s,$$

for P an appropriate permutation matrix. (This is Pólya's proof.) An exhaustive treatment of the inequalities connecting eigenvalues and singular values is found in [7], [8], [11], [1].

5. Other functions on Ω_n and the permanent problem. In 1926 B. L. van der Waerden [29] proposed the following problem: let $p(S)$ denote the *permanent* of S,

$$p(S) = \sum_{\sigma} \prod_{i=1}^{n} s_{i\sigma(i)},$$

where the sum is over all permutations σ of $1, \cdots, n$; question: is the minimum value of this function on Ω_n assumed uniquely for $S=J_n$, the d.s. matrix all of whose entries are $1/n$? For $n=2$ the problem is trivial and for $n=3$ the answer is "yes." In addition to this it is known that if the minimum occurs in the relative interior of Ω_n (*i.e.*, d.s. matrices with positive entries) then it occurs uniquely at J_n [20]. P. Erdös observed that two consequences of the van der Waerden conjecture are: (a) for any $S\in\Omega_n$ there exists a permutation σ such that $\prod_{i=1}^{n} s_{i\sigma(i)} \geqq (1/n)^n$; (b) there exists a permutation σ such that $\sum_{i=1}^{n} s_{i\sigma(i)} \geqq 1$ and $s_{i\sigma(i)}>0$. Conjecture (a) seems about as hard as the original problem but (b) appears as a corollary of a considerably stronger result [21]:

If $S\in\Omega_n$ and has k eigenvalues of absolute value 1 (1 *itself is always an eigenvalue*) *then there exists a permutation σ such that $s_{i\sigma(i)}>0$, $i=1, \cdots, n$ and $\sum_{i=1}^{n} s_{i\sigma(i)} \geqq k$. The case of equality can be removed unless S can be brought to the form*

$$J_{n_1} + \cdots + J_{n_k}, \qquad \sum_{j=1}^{k} n_j = n \tag{5.1}$$

by permutation of rows and columns.

Let the p-minor of s_{ij} be the permanent of the $(n-1)$-square matrix obtained by deleting the ith row and jth column of S. The following question has arisen in connection with the conjecture of van der Waerden; is it possible for some $S\in\Omega_n$ that

(i) the p-minors of all positive entries of S are equal with common value $p(S)$,

(ii) the p-minors of all zero entries of S are equal with common value $p(S)+\beta$ for $\beta>0$?

If the answer to this is "no" for general n (which is the case for $n=3$ and 4) then the permanent problem will be solved. The separation of p-minors into two equal sets can occur of course, but apparently not for $\beta>0$:

$$S = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

6. Kakutani's conjecture. A partial ordering can be introduced in Ω_n as fol-

lows: $A<B$ if there exists $C\in\Omega_n$ for which $A=CB$. A partial ordering of real n-tuples can also be defined by: $a<b$ if and only if for every convex function ϕ

$$\sum_{i=1}^{n}\phi(a_i)\leqq\sum_{i=1}^{n}\phi(b_i).$$

A result in [9] states that $a<b$ *if and only if* $a\in\Omega_n(b)$. Thus if $a<b$, $a=Sb$ for some $S\in\Omega_n$. Now Kakutani conjectured that *if* $Aa<Ba$ *for all real* n*-tuples* a, *then* $A<B$. The result holds for $n=3$ but A. Horn constructed A and B for $n=4$ for which it fails [27]:

$$A=\begin{pmatrix}\frac{1}{4}&0&\frac{3}{8}&\frac{3}{8}\\ \frac{1}{4}&0&\frac{3}{8}&\frac{3}{8}\\ \frac{1}{4}&\frac{1}{2}&\frac{5}{32}&\frac{3}{32}\\ \frac{1}{4}&\frac{1}{2}&\frac{3}{32}&\frac{5}{32}\end{pmatrix},\qquad B=\begin{pmatrix}\frac{1}{4}&0&\frac{1}{4}&\frac{1}{2}\\ \frac{1}{4}&0&\frac{1}{2}&\frac{1}{4}\\ \frac{1}{4}&\frac{3}{4}&0&0\\ \frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\end{pmatrix}.$$

However in case B is nonsingular an argument of S. Sherman [28] will establish the result. Recently S. Schreiber [26] gave a simpler argument for B nonsingular as well as geometric interpretation of this partial ordering in Ω_n.

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CONVERGENCE IN ORDERED INTEGRAL DOMAINS

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1. Introduction. All the terms used in defining convergence of a sequence of real numbers have meaning in any ordered integral domain. Accordingly, we say: a sequence of elements $\{a_n\}$ of an ordered integral domain D is convergent if there exists an element a with the property that for every element $\epsilon>0$ there is an integer N such that $|a_n-a|<\epsilon$ for $n>N$. (The double use of the inequality sign, to denote inequalities in D and in the domain of ordinary integers, will cause no confusion.) We write, as usual, $a_n\to a$.

If D is archimedean ordered (*i.e.*, whenever $a>0$, $b>0$ there exists an integer n such that $na>b$) it is order-equivalent to a subdomain of the real numbers, and all the usual questions concerning convergence in D have the usual answers.

The following example illustrates what may happen if the ordering of D is not archimedean.

Let D consist of all real numbers expressible in the form $\sum_{i=0}^{k} \alpha_i \pi^i$, $k \geqq 0$, the α_i rational. Call this element positive (or negative) if α_k (the last non zero coefficient) is positive (or negative) in the usual sense. In this domain: (a) if $a_n \to \pi$, $b_n \to 0$ and $b_n \neq 0$ for every n, then $\{a_n b_n\}$ does not converge, (b) if $a_n = \pi + 1/n$, $b_n = \pi - 1/n$, then $a_n b_n \to \pi^2$, but $\{a_n b_{n+1}\}$ does not converge.

2. Preliminaries. If the ordered integral domain D has a smallest positive element ϵ and $a_n \to a$, then $|a_n - a| < \epsilon$ for $n > N$ implies $a_n = a$ for all such n; thus, $\{a_n\}$ can converge if and only if all a_n are equal for n sufficiently large. (We shall call such a sequence trivial.) Since this situation is without interest, we suppose henceforth that D has no smallest positive element.

The following results are easily established:

(1) A convergent sequence has a unique limit.

(2) A convergent sequence has the Cauchy property (*i.e.*, if $a_n \to a$ then for every $\epsilon > 0$ there is an N such that $|a_n - a_m| < \epsilon$ when $n > N$, $m > N$.

(3) If $a_n \to a$, $b_n \to b$ then $a_n \pm b_n \to a \pm b$.

(4) Every subsequence of a convergent sequence converges to the same limit as the sequence.

For example, to establish (1), suppose $a_n \to a$, $a_n \to b$. Let $\epsilon > 0$. Let δ be chosen so that $0 < \delta < \epsilon$. Let $|a_n - a| < \delta$ for $n > N$ and $|a_n - b| < \epsilon - \delta$ for $n > N'$. Then, for $n > \max(N, N')$, $|a - b| \leqq |a - a_n| + |a_n - b| < \epsilon$. Hence $a - b = 0$.

3. Bounded elements. We call an element a of D bounded if there exists an element $\alpha > 0$ such that $\alpha|a| \leqq 1$. If a and b are bounded, let $\alpha|a| \leqq 1$, $\beta|b| \leqq 1$ where $\alpha > 0$, $\beta > 0$. If $\gamma = \min(\alpha, \beta)$, then $\gamma|a \pm b| \leqq \gamma|a| + \gamma|b| \leqq \alpha|a| + \beta|b| \leqq 2$. Let $0 < \delta < 1$ and $\epsilon = \min(\delta, 1 - \delta)$. Since $2\delta + 2(1 - \delta) = 2$, therefore either $2\delta \leqq 1$ or $2(1 - \delta) \leqq 1$. Hence $2\epsilon \leqq 1$. Therefore $\epsilon\gamma|a \pm b| \leqq 2\epsilon \leqq 1$. Thus, $a \pm b$ are bounded. Also, since $\alpha\beta|ab| \leqq 1$, ab is bounded.

Since 1 is a bounded element and the sum, difference and product of two bounded elements are bounded, the bounded elements of D form a subdomain. Since this subdomain contains 1 it contains every integral element; hence every integral element of D is bounded.

If $|a| \leqq b$ and b is bounded, clearly a is also bounded. Hence every element of an archimedean ordered domain is bounded, since for every a there is an integral element n such that $|a| \leqq n$.

If a is bounded and $\alpha|a| \leqq 1$ where $\alpha > 0$, then for every $\epsilon > 0$ we have $(\alpha\epsilon)|a| \leqq \epsilon$. Thus, there exists a $\delta > 0$ such that $\delta|a| \leqq \epsilon$. Conversely, if for a given element a there exists for every $\epsilon > 0$ a $\delta > 0$ such that $\delta|a| \leqq \epsilon$ then (taking $\epsilon = 1$) a is bounded. It is this property of bounded elements that is used so frequently in proofs of convergence theorems in the real domain.

4. Product sequences. The main difference between convergence in D and

convergence in the real domain occurs in connection with the product sequence $\{a_nb_n\}$ of $\{a_n\}$ and $\{b_n\}$.

Let $a_n \to a$, $b_n \to b$, so that $a_n = a+\alpha_n$, $b_n = b+\beta_n$ where $\alpha_n \to 0$, $\beta_n \to 0$. Then $a_nb_n = ab+(a\beta_n+b\alpha_n)+\alpha_n\beta_n$. Since $|\alpha_n| < 1$ for n sufficiently great and $\beta_n \to 0$, therefore $\alpha_n\beta_n \to 0$.

Case 1. *If* $\{a_n\}$, $\{b_n\}$ *are both trivial, then obviously* $a_nb_n \to ab$.

Case 2. *If* $\{a_n\}$ *is trivial and* $\{b_n\}$ *is not, then* $\{a_nb_n\}$ *converges if and only if* a *is bounded, in which case* $a_nb_n \to ab$.

Proof. Since $\{a_n\}$ is trivial, $a_n = a$ for all sufficiently large n. Suppose $\{a_nb_n\}$ converges. Then for m and n sufficiently large $|a_nb_n - a_mb_m| = |a|\,|b_n - b_m| < 1$. Since $\{b_n\}$ is nontrivial, there exist such m and n for which $b_n - b_m \neq 0$. Thus, a is bounded.

Conversely, suppose a is bounded. Let $\epsilon > 0$. Let δ be a positive element so chosen that $\delta|a| \leqq \epsilon$. Let N be chosen so that $|\beta_n| < \delta$ for $n > N$, which can be done since $\beta_n \to 0$. Then $|a\beta_n| \leqq \delta|a| \leqq \epsilon$ for $n > N$. Hence $a\beta_n \to 0$. Since $\alpha_n = 0$ for n sufficiently great, therefore $b\alpha_n \to 0$. Thus, $a_nb_n \to ab$.

Case 3. *If neither* $\{a_n\}$ *nor* $\{b_n\}$ *is trivial and* a *is bounded, then* $\{a_nb_n\}$ *converges if and only if* b *is bounded, in which case* $a_nb_n \to ab$.

Proof. Since a is bounded, Case 2 implies $a\beta_n \to 0$. If b is bounded then, by Case 2, $b\alpha_n \to 0$. Hence $a_nb_n \to ab$. Conversely, if $a_nb_n \to c$ then $b\alpha_n = a_nb_n - ab - a\beta_n - \alpha_n\beta_n \to c - ab$. Hence, by Case 2, b is bounded.

Case 4. *If neither* $\{a_n\}$ *nor* $\{b_n\}$ *is trivial and neither* a *nor* b *is bounded, then* $\{a_nb_n\}$ *converges if and only if there exist elements* $\alpha \neq 0$, $\beta \neq 0$ *such that* $a\alpha+b\beta$ *is bounded and* $\alpha\alpha_n = \beta\beta_n$ *for all* n *sufficiently great, in which case* $a_nb_n \to ab$.

Proof. (a) Suppose $a_nb_n \to c$. Then, letting $\gamma_n = a\beta_n + b\alpha_n$, we have $\gamma_n = a_nb_n - ab - \alpha_n\beta_n \to c - ab$. Let N be so chosen that for $n > N$, $m > N$

$$|\gamma_n - \gamma_m| = |a(\beta_n - \beta_m) + b(\alpha_n - \alpha_m)| < 1, \quad |\alpha_n - \alpha_m| < 1, \ |\beta_n - \beta_m| < 1.$$

Let i, j be integers so chosen that $i > N$, $j > N$, $|\beta_i - \beta_j| \neq 0$. Let $\alpha = \beta_i - \beta_j$, $\beta = \alpha_i - \alpha_j$. Then $0 < |\alpha| < 1$, $|\beta| < 1$. Also $|a\alpha+b\beta| = |\gamma_i - \gamma_j| < 1$, so that, since a is not bounded, $\beta = 0$. Let $\gamma = a\alpha+b\beta$. By Cases 1 and 2, we have $\beta\gamma_n \to \beta(c-ab)$ and $\gamma\alpha_n \to 0$, since β and γ are bounded. Also, $\beta\gamma_n = a\beta_n\beta + b\alpha_n\beta = a(\beta_n\beta - \alpha_n\alpha) + \gamma\alpha_n$, so that $a(\beta_n\beta - \alpha_n\alpha) = \beta\gamma_n - \gamma\alpha_n \to \beta(c-ab)$. But $\beta_n\beta - \alpha_n\alpha \to 0$ since $\beta_n\beta \to 0$ and $\alpha_n\alpha \to 0$. Since $\{a(\beta_n\beta - \alpha_n\alpha)\}$ is convergent and a is not bounded, Case 2 implies that $\{\beta_n\beta - \alpha_n\alpha\}$ is a trivial sequence. Thus, $\alpha_n\alpha = \beta_n\beta$ for all n sufficiently great. And, since $\beta(c-ab) = 0$, we have $c = ab$.

(b) Conversely, suppose there exist elements α and β with the properties stated above. Then $\alpha(a\beta_n + b\alpha_n) = (\alpha a + \beta b)\beta_n$ for n sufficiently great, since $\alpha\alpha_n = \beta\beta_n$. By Case 2, since $\beta_n \to 0$ and $\alpha a + \beta b$ is bounded, $(\alpha a + \beta b)\beta_n \to 0$. Therefore also $\alpha(a\beta_n + b\alpha_n) \to 0$. Hence $a\beta_n + b\alpha_n \to 0$ (for otherwise for some $\epsilon > 0$ we

would have $|a\beta_n+b\alpha_n| \geqq \epsilon$ for infinitely many values of n; for these values of n we would have $|\alpha(a\beta_n+b\alpha_n)| \geqq |\alpha|\epsilon>0$, which is impossible). Thus, $a_nb_n \to ab$.

Remark. If $a_n \to a$ and a is not bounded, for a given b there is "at most one" sequence $\{b_n\}$ converging to b such that $\{a_nb_n\}$ is convergent. That is, if $b_n \to b$, $b_n' \to b$, and $\{a_nb_n\}$, $\{a_nb_n'\}$ both converge, then $b_n = b_n'$ for all n sufficiently great. For, if both these product sequences converge, then $a_nb_n - a_nb_n' = a_n(b_n - b_n') \to 0$. But $b_n - b_n' \to 0$ and 0 is a bounded element, so that if $\{b_n - b_n'\}$ were not a trivial sequence then $\{a_n(b_n - b_n')\}$ could not converge (by Case 2 if $\{a_n\}$ is trivial and by Case 3 if it is not). Hence $\{b_n - b_n'\}$ must be a trivial sequence.

5. Quotient sequences. Let $a_n \to a$, $b_n \to b$ and suppose $c_n = a_n/b_n$ exists for every n. If $c_n \to c$ then, since $a_n = b_nc_n \to a$, we have $a = bc$. Thus a/b exists and $c = a/b$.

Assuming a/b exists, we state without proof the following conditions for the convergence of $\{c_n\}$:

Case 1. $\{c_n\}$ *is trivial (in which case it is convergent) if and only if* $ab_n = ba_n$ *for all* n *sufficiently great.*

Case 2. *If* $\{c_n\}$ *is nontrivial and* a/b *is bounded, then* $\{c_n\}$ *is convergent if and only if* b *is bounded.*

Case 3. *If* $\{c_n\}$ *is nontrivial and* a/b *is not bounded, then* $\{c_n\}$ *converges if and only if either* (a) $\{b_n\}$ *is trivial or* (b) $\{b_n\}$ *is nontrivial,* b *is not bounded, and there exist elements* $\alpha \neq 0$, $\beta \neq 0$ *such that* β *is bounded and* $\alpha(a_n - a) = (\alpha(a/b)+\beta b_n)(b_n - b)$ *for all* n *sufficiently great.*

Remark. If $\{a_n\}$ and $\{b_n\}$ are nontrivial and converge to limits a and b which are not bounded, it is impossible for both the product and quotient sequences to converge. In fact, if $\{a_nb_n\}$ converges, it is impossible for both a/b and a_n/b_n (for all n sufficiently large) to exist.

For, suppose $\{a_nb_n\}$ converges and a/b, a_n/b_n exist for $n > N$. Then (referring to Case 4 of the preceding section) $\gamma = a\alpha+b\beta$ is bounded and divisible by b. Let $\gamma = bd$. Since γ is bounded, there exists an element $x>0$ such that $(x|d|)|b| = x|\gamma| \leqq 1$. Since b is not bounded, this requires $x|d| = 0$. Hence $d = 0$, so that $\gamma = 0$. Thus $a\alpha = -b\beta$.

Since $\alpha\alpha_n = \beta\beta_n$ for all n sufficiently large, we have, for these values of n, $\alpha(a_n - a) = \beta(b_n - b)$, $\alpha a_n - \beta b_n = \alpha a - \beta b = -2\beta b$; hence $2\beta b$ is divisible by b_n when n is large enough. Since $2\beta\beta_n = 2\beta(b_n - b) = 2\beta b_n - 2\beta b$, therefore $2\beta\beta_n$ is also divisible by b_n.

For n sufficiently large, $|\beta_n| < 1$, so that β_n is bounded. Reference to the proof of Case 4 in the preceding section shows $0 < |\beta| < 1$, so that β is bounded. Thus, $2\beta\beta_n$ is bounded and nonzero so that, since b_n is one of its factors, b_n is also bounded. Since $|b| \leqq |b_n| + |b - b_n| \leqq |b_n| + 1$ for n sufficiently large, b is bounded, which is a contradiction of the hypothesis.

6. Quotient fields. Let F be the quotient field of D ordered so as to preserve the ordering of D. Let $a_n \to a$ nontrivially in D. Suppose D contains an element b which is not bounded. Since $\{a_n\}$ is a nontrivial sequence, therefore for infinitely many values of n, $|a_{n+1}-a_n|>0$. Since b is not bounded, for these values of n we have $|a_{n+1}-a_n|\,|b|>1$. Thus, in F, $|a_{n+1}-a_n|>1/|b|$. Hence $\{a_n\}$ cannot converge in F since it does not have the Cauchy property in F.

Suppose, on the other hand, that every element of D is bounded. If $\epsilon>0$ in F, let $\epsilon=\alpha/\beta$ where α and β are positive elements of D. In D, $a_n-a\to 0$, so that $\beta(a_n-a)\to 0$. Hence there exists an N such that $|\beta(a_n-a)|<\alpha$ for $n>N$. Thus, for $n>N$, $|a_n-a|<\alpha/\beta=\epsilon$. Therefore $a_n\to a$ in F.

7. Sequence $\{a^n\}$. There are integral domains in which $\{a^n\}$ does not converge for any $a\neq 0, 1$. Suppose, however, that it does converge, say to b. If the sequence is trivial then $a^n=a^m$, $m>n$, for sufficiently great n, so that $a^{m-n}=1$. Hence $a=0$, 1, or -1. Clearly for $a=0$, 1 we have convergence and for $a=-1$ we do not.

Suppose now that $a^n\to b$ and $\{a^n\}$ is not trivial. The sequence $\{aa^n\}$ may be regarded as a subsequence of $\{a^n\}$, and thus convergent to b, and as the product sequence of $\{a^n\}$ and $\{a\}$ and thus, since it does converge, convergent to ab. Hence $ab=b$, so that $b=0$. Also, since $|a^n|<1$ for n sufficiently large, $|a|<1$.

To consider the converse problem, let D' denote the subdomain consisting of all the bounded elements of D, ordered according to the ordering of D. If $0<a<1$ then a is in D'; if $a^n\to 0$ in D then $a^n\to 0$ in D', and conversely.

We now show: if and only if D' is archimedean ordered does $0<a<1$ imply $a^n\to 0$.

(a) We show first: if D' contains an element $b>0$ such $b^n\to 0$ and $Nb\geqq 1$ for some integer N, then D' is archimedean ordered. Let ϵ, δ be positive elements of D'. Since ϵ is a bounded element of D, there is an element $\alpha>0$ in D such that $0<\alpha\epsilon\leqq 1$. Since $b^n\to 0$ there exists a K such that $b^n<\alpha\delta$ for $n>K$. Since $Nb\geqq 1$, we have $1\leqq N^nb^n<N^n\alpha\delta$ for $n>K$. Hence, $\epsilon<N^n\alpha\delta\epsilon\leqq N^n\delta$. Thus $N^n\delta>\epsilon$, and the desired result is established since N^n is an integer.

Now, suppose D contains an element a between 0 and 1 such that $\{a^n\}$ and $\{(1-a)^n\}$ both converge. Since either $2a\geqq 1$ or $2(1-a)\geqq 1$, it follows from the preceding, with $N=2$, that D' is archimedean ordered.

(b) Conversely, suppose D' is archimedean ordered. Let $0<a<1$. Let k be a positive integer so chosen that $k(1-a)>a$. (This can be done since $1-a$ is in D'). By induction on n it follows that $na^n\leqq ka$ for all positive integral n. (For, if $na^n\leqq ka$, then $(n+1)a^{n+1}=(na^n+a^n)a\leqq(ka+a)a\leqq ka$). Let ϵ be a positive element of D' and let integer N be so chosen that $N\epsilon>a$. Then, for $n>kN$, $kNa^n<na^n\leqq ka<kN\epsilon$, from which $a^n<\epsilon$. Thus, $a^n\to 0$.

8. Sequence $\{a/n\}$. Suppose a is a positive element of D divisible by every positive integral element n. If $a/n\to b$, then $a/(n(n+1))=(a/n)-(a/(n+1))\to 0$. But $\{a/(n(n+1))\}$ is a subsequence of $\{a/n\}$ and, therefore, converges to b.

Hence $b=0$. Therefore, also, a is a bounded element, since $a/n<1$, or $a<n$, for n sufficiently great. If, in addition, $Na\geqq 1$ for some integer N, then D', the subring of bounded elements, must be archimedean. For, if δ is a positive element of D' then $N\delta a/n\to 0$, since $N\delta$ is a bounded element. Therefore, if ϵ is any positive element of D', we have $\delta/n\leqq N\delta a/n<\epsilon$ for n sufficiently great, so that $\delta<n\epsilon$.

Conversely, if D' is archimedean ordered and a is bounded then $a/n\to 0$. For, if ϵ is a positive element of D' and K is an integer so chosen that $K\epsilon>a$, then $a/n<\epsilon$ for $n>K$.

9. Limitability. In D let $a_n\to a$ and suppose $S_n=(a_1+\cdots+a_n)/n$ exists for $n=1, 2, \cdots$. In the real domain $\{S_n\}$ also converges to a, which shows that a convergent sequence is limitable by Cesàro or Hölder means of the first order to the same number as the limit of the sequence. For a general integral domain, even a field, this may not be true. For instance, in a non-archimedean ordered field if $a_1=1$, $a_n=0$ for $n>1$, then $a_n\to 0$ but $\{S_n\}=\{1/n\}$ does not converge.

However, we now show, if $a_n\to a$ in D and $S_n\to b$, then $b=a$. Let $a_n=a+\alpha_n$. Then $\alpha_n\to 0$. Also $S_n=a+T_n$, where $T_n=(\alpha_1+\cdots+\alpha_n)/n$. Since $a_n\to a$ and $S_n\to b$, $T_n\to b-a$. Hence $T_n=b-a+\beta_n$, where $\beta_n\to 0$. We have

$$n(\beta_n-\beta_{n-1})=n(T_n-T_{n-1})=n\left(\frac{\alpha_1+\cdots+\alpha_n}{n}-\frac{\alpha_1+\cdots+\alpha_{n-1}}{n-1}\right)$$

$$=\alpha_n+(\alpha_1+\cdots+\alpha_{n-1})\left(1-\frac{n}{n-1}\right)=\alpha_n-T_{n-1}.$$

Hence $n(\beta_n-\beta_{n-1})\to a-b$.

Suppose $a-b>0$. (This involves no loss of generality since if $a-b<0$ we may let $\gamma_n=-\beta_n$ and we have $\gamma_n\to 0$, $n(\gamma_n-\gamma_{n-1})\to b-a>0$). Choose α and γ so that $0<\gamma<\alpha<a-b$. Let $|n(\beta_n-\beta_{n-1})-(a-b)|<(a-b)-\alpha$ for $n>N'$. Then $\alpha-(a-b)<n(\beta_n-\beta_{n-1})-(a-b)<(a-b)-\alpha$. Hence $n(\beta_n-\beta_{n-1})>\alpha$ for $n>N'$.

Let $|\beta_n|<\gamma$ for $n>N''$. Let $N>\max(N', N'')$ and let N''' be so chosen that

$$\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{n}>1 \qquad \text{for } n>N'''.$$

(Here we are dealing with ordinary rational numbers). By induction on n, making use of $n(\beta_n-\beta_{n-1})>\alpha$, we have, for $n>N$,

$$n(n-1)\cdots(N+1)(\beta_n-\beta_N)$$

$$>\alpha\left[\frac{n(n-1)\cdots(N+1)}{N+1}+\frac{n(n-1)\cdots(N+1)}{N+2}+\cdots\right.$$

$$\left.+\frac{n(n-1)\cdots(N+1)}{n}\right].$$

From this it follows that, for $n>\max(N, N''')$,

$$n(n-1)\cdots(N+1)(\beta_n-\beta_N)>\alpha n(n-1)\cdots(N+1),$$

so that $\beta_n-\beta_N>\alpha$ or $\beta_n>\alpha+\beta_N$.

Since $n(\beta_n-\beta_{n-1})>\alpha>0$, for $n>N'$, $\beta_{n-1}<\beta_n$. Since $\beta_n\to 0$, this requires that $\beta_n<0$ for $n>N'$. Thus, $-\beta_N=|\beta_N|<\gamma$, so that $\beta_N+\alpha=\gamma+\beta_N+(\alpha-\gamma)>\alpha-\gamma>0$. Hence, for $n>\max(N, N''')$ we have $\beta_n>\alpha+\beta_N>\alpha-\gamma>0$, contradicting the fact that $\beta_n\to 0$. Thus $a-b>0$ is impossible and our assertion that $a=b$ is proved.

We now show:

(a) If D is archimedean ordered and $a_n\to a$, then $S_n\to a$ provided the S_n all exist. In the notation above, let $\epsilon>0$ and suppose $|\alpha_n|<\epsilon$ for $n>M$. Let K be an integer, so chosen that $|\alpha_1+\cdots+\alpha_M|<K\epsilon$. For $n>\max(M, K)$ we have

$$\begin{aligned}|nT_n| &= |\alpha_1+\cdots+\alpha_M+\alpha_{M+1}+\cdots+\alpha_n| \\ &\leqq |\alpha_1+\cdots+\alpha_M|+|\alpha_{M+1}|+\cdots+|\alpha_n| \\ &< K\epsilon+(n-M)\epsilon<2n\epsilon,\end{aligned}$$

so that $|T_n|<2\epsilon$. Let $\delta>0$ and let η be so chosen that $0<\eta<\delta$. Then either $2\eta\leqq\delta$ or $2(\delta-\eta)\leqq\delta$. Hence, with the proper choice of ϵ, we have $|T_n|<2\epsilon\leqq\delta$ for n sufficiently large, so that $T_n\to 0$ and $S_n\to a$.

(b) Suppose D contains a positive element a divisible by every positive integral element and such that $Ma\geqq 1$ for some integer M. Then, if every convergent sequence (or merely every trivial sequence) is limitable by first order Cesàro means, D is archimedean ordered.

For, if b is any element of D then, letting $a_1=ab$, $a_n=0$ for $n>1$, we have $a_n\to 0$, $S_n=ab/n\to 0$. Thus, $(a/n)|b|<1$ for some n, so that b is bounded. Thus D', the subring of bounded elements, is D itself. Since (taking $b=1$) $\{a/n\}$ is convergent and $Ma\geqq 1$, by the preceding section, D is archimedean ordered.

10. Domains with valuations. Let I be an integral domain with a valuation function $\phi(x)$ whose values lie in the ordered integral domain D. That is, (1) for each x in I, $\phi(x)$ is a unique element in D; (2) $\phi(0)=0$; (3) $\phi(x)>0$ if $x\neq 0$, (4) $\phi(xy)=\phi(x)\phi(y)$; (5) $\phi(x+y)\leqq\phi(x)+\phi(y)$.

If $\{x_n\}$ is a sequence of elements in I, we say it is convergent (relative to the valuation function) if there exists an element x in I such that $\phi(x_n-x)\to 0$ in D. We call an element x in I bounded if $\phi(x)$ is a bounded element in D. All of the above results for convergence in D can easily be restated to apply to convergence in I relative to $\phi(x)$.

A GENERALIZED INDEPENDENCE CONDITION AND ERROR CORRECTION CODES*

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The notion of independence of a set of vectors is fundamental to the theory of vector spaces. In this note we define a generalization of this concept for vector spaces over finite fields and present an application to the theory of error correction codes. Before proceeding to this generalization we shall digress to present a few facts concerning error correction codes.

Let M be a finite metric space and C a subset of M such that elements in C are separated from one another by at least d_0 units. Such a subset will be called a code. In order to motivate this terminology consider a communication system in which the transmitting station selects its messages from C. If the transmission channel linking the transmitting and receiving stations is noiseless there will be no difficulty in recovering the transmitted message. If noise is present let us assume that the received message is an element of M which is "close" to the transmitted message. If, in fact, the distance a transmitted message moves may be bounded by $\frac{1}{2}d_0-\epsilon$ (for some $\epsilon>0$) there is an element of C, namely the transmitted message, which is closest to the received message. Decoding, or the recovery of the transmitted message, is accomplished by finding that (unique) element of C closest to the received message. We say that information has been encoded by restricting the class of transmittable messages to C and thus removing or lessening the chance of ambiguity arising due to noise.

Now suppose that M consists of all 2^n sequences of n symbols, each symbol assuming one of two states which we denote by zero and one. If the elements of M are identified with the vertices of a unit-edge cube in E^n we define the distance between vertices to be the minimum number of edges that must be traversed in going from one vertex to the other. An error is said to have occurred in the ith symbol if the transmitted and received ith symbols do not agree. Errors are introduced by the channel performing a chance experiment on each of the n symbols; the ith symbol is received as transmitted with probability p and altered with probability $1-p$. It is assumed that these chance experiments are independent and have the same probability p associated with them. If $d_0=2e+1$ and if no more than e alterations occur in the transmission of a message then there is a unique element of C closest to the received message and this is the most probable transmitted message. In this case C is an e-order error correction code. A central problem is the determination of the largest code C with a fixed n and e. Such error correction codes have been studied by Hamming [1], Slepian [2] and Reed [3].

DEFINITION 1. *Let V be a finite-dimensional vector space over a finite field F. A set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is k-independent if every subset of k of these vectors*

* The work described here was made possible by support extended by the Air Force Cambridge Research Center on contract AF19(604)-1836.

is independent. The set is k-dependent if some subset of k of these vectors is dependent.

DEFINITION 2. *With V and F as in Definition* 1, *a set of vectors* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *is k-full if the set is k-independent and not the proper subset of any k-independent set. A set is k-maximal if, among all k-full sets, it has a maximum number of elements.*

Let $C^1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and denote by C^m the set of all elements of the form

$$\sum_{i=1}^{m} a_i \mathbf{v}_{j_i},\ a_i \neq 0,\ a_i \in F,\ i = 1, \cdots, m,\ 1 \leqq j_1 \cdots < j_m \leqq n. \tag{1}$$

Let $C^0 = \{\mathbf{0}\}$, the null vector in V.

A k-full set plays the role of a basis in V; to show this we prove the

LEMMA. *Let* C^1 *be k-independent. Then* C^1 *is k-full if and only if* $\bigcup_{i=0}^{k-1} C^i = V$.

Proof. If C^1 is k-full and $\bigcup_{i=1}^{k-1} C^i \subset V$, choose $\mathbf{v}$ in $V - \bigcup_{i=0}^{k-1} C^i$ and let $D^1 = C^1 \cup \{\mathbf{v}\}$. Now

$$\mathbf{v} + \sum_{i=1}^{m} a_i \mathbf{v}_{j_i} \neq \mathbf{0} \tag{2}$$

for any choice of the a_i and $m \leqq k-1$, since equality would imply $\mathbf{v} \in \bigcup_{i=0}^{k-1} C^i$. But (2) implies that D^1 is k-independent, a contradiction of the hypothesis. Conversely, if $\bigcup_{i=0}^{k-1} C^i = V$, then given $\mathbf{v} \in V$ we have a (not necessarily unique) representation

$$\mathbf{v} = \sum_{i=1}^{m} a_i \mathbf{v}_{j_i} \tag{3}$$

for some choice of the a_i and $m \leqq k-1$. Equation (3) shows that $\mathbf{v}$ may not be added to C^1 without $C^1 \cup \{\mathbf{v}\}$ becoming k-dependent. Thus C^1 is k-full and the lemma is proved.

Let J_2 be the field $\{0, 1\}$ and $\Omega^n = J_2 \times \cdots \times J_2$ (n copies). Ω^n is an n-dimensional vector space over J_2 with addition defined in the following manner: If

$$\mathbf{v} = (a_1 \cdots a_n) = (a_i), \qquad \mathbf{w} = (b_1 \cdots b_n) = (b_i), \tag{4}$$

then

$$\mathbf{v} + \mathbf{w} = (c_i), \qquad c_i = a_i \oplus b_i, \tag{5}$$

where $\oplus$ denotes addition modulo 2.

Following Hamming [1], we define a metric d by

$$d(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} (a_i \oplus b_i). \tag{6}$$

DEFINITION 3. *A subset C of Ω^n is a k-code if* $\mathbf{v}, \mathbf{w} \in C$, $\mathbf{v} \neq \mathbf{w}$, *imply* $d(\mathbf{v}, \mathbf{w}) \geqq k+1$.

If it is further assumed that C is a subspace of Ω^n (of dimension m, $1 \leqq m \leqq n$), then Slepian [2] has shown that the elements of C are of the form (up to a permutation of the coordinates)

$$(\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_r), \qquad m + r = n, \tag{7}$$

$$\beta_j = \sum_{i=1}^{m} \gamma_{ij}\alpha_i \pmod 2, \qquad \gamma_{ij} \in J_2, j = 1, \cdots, r. \tag{8}$$

In this case C is said to be a *systematic k-code*. For such a code let

$$\mathbf{v}_i = (\gamma_{i1} \cdots \gamma_{ir}), \qquad i = 1, \cdots, m; \tag{9}$$

$$\mathbf{v}_{m+i} = (\underbrace{0 \cdots 0}_{i-1\ 0\text{'s}}\ 1\ \underbrace{0 \cdots 0}_{r-i\ 0\text{'s}}), \qquad i = 1, \cdots, r; \tag{10}$$

$$C^1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}. \tag{11}$$

The following equivalence is to be proved:

THEOREM. *A necessary and sufficient condition that a subspace C (defined by* (7) *and* (8)) *be a systematic k-code is that the set C^1 of* (11) *be k-independent.*

Proof. Let C be a systematic k-code and suppose that for some p indices $1 \leqq j_1 < \cdots < j_p \leqq n$ we have

$$\mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = \mathbf{0}. \tag{12}$$

Let R be the mapping

$$R\colon \Omega^m \to \Omega^r, \qquad R((\alpha_1 \cdots \alpha_m)) = (\beta_1 \cdots \beta_r), \tag{13}$$

where the β_j are given by (8). Let f be the mapping

$$\begin{gathered} f\colon \Omega^n \to \Omega^r, \\ f((u_1 \cdots u_n)) = R((u_1 \cdots u_m)) + (u_{m+1} \cdots u_n). \end{gathered} \tag{14}$$

f is a linear transformation of Ω^n onto Ω^r with kernel C. Next, let

$$\mathbf{x}_i = (\underbrace{0 \cdots 0}_{i-1\ 0\text{'s}}\ 1\ \underbrace{0 \cdots 0}_{n-i\ 0\text{'s}}), \qquad i = 1, \cdots, n. \tag{15}$$

Equations (7)–(8) and (13)–(14) show that $\mathbf{v}_i = f(\mathbf{x}_i) \in C^1$, $i = 1, \cdots, n$. Let $\mathbf{x}$ be an arbitrary element in C and

$$\mathbf{y} = \mathbf{x} + (\mathbf{x}_{j_1} + \cdots + \mathbf{x}_{j_p}). \tag{16}$$

Since $\mathbf{x}$ is in C, $f(\mathbf{x}) = \mathbf{0}$ and thus $f(\mathbf{y}) = \mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = \mathbf{0}$ by (12). Thus $\mathbf{y}$ is in C and $d(\mathbf{x}, \mathbf{y}) = p$. Since C is a k-code, $p \geqq k+1$ and we conclude that C^1 is k-independent.

Now assume that C^1 is k-independent and choose $\mathbf{x}$ and $\mathbf{y}$ to be distinct elements in C. Then for some p and choice of indices,

$$\mathbf{x} = \mathbf{y} + (\mathbf{x}_{j_1} + \cdots + \mathbf{x}_{j_p}). \tag{17}$$

Since $f(\mathbf{x}) = f(\mathbf{y}) = \mathbf{0}$, we have $\mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = 0$. But $p \geqq k+1$ since C^1 is k-independent and $d(\mathbf{x}, \mathbf{y}) = p$, from which we conclude that C is a k-code.

By this theorem, the construction of (binary) error correction codes is equivalent to the generation of k-independent sets. At present there are procedures for generating such full sets, but no method is known for the systematic construction of k-maximal sets.

The author wishes to thank Professor J. Wolfowitz for his helpful suggestions.

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SOME PACKING PROBLEMS

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How many queens may be packed on an ordinary chess board without any pair challenging each other? How many rooks, kings, bishops or horses?

A little experimenting soon leads to the suspicion that the maxima are (with a fairly obvious notation) $Q=8$, $R=8$, $K=16$, $B=14$, $H=32$. These scores may be achieved in various ways, for example, as shown in Figure 1.

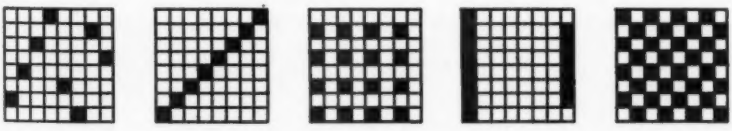


FIG. 1

The demonstration that these packings are best possible is quite easy and involves only the most elementary methods.

In the case of rooks or queens, more than eight would certainly require that two or more share a row and hence challenge. For bishops it is enough to count the parallel diagonals (fifteen) and observe that both extremes (one square long) cannot be used simultaneously. The maximum of 32 for horses would follow at once if a maximum of 8 could be established for a 4×4 chessboard. But if more than 8 could be packed on this smaller board then at least one of the 2×2 corners would contain 3 or 4 horses. That this is impossible is easily discovered by an examination of the few possible cases. The result for kings follows in precisely the same way.

Much more interesting is the generalization to an $N\times N$ board. The same arguments used for $N=8$ suffice in the general case when rooks or bishops are concerned. With an obvious extension of our notation and $N>1$,

$$R_N = N, \qquad B_N = 2N - 2.$$

The case of queens is more elusive. Since only one queen may occur in each row or column, it is convenient to describe a solution as a permutation of the integers $1, \cdots, N$. For each column from left to right we merely indicate the row occupied. The above solution for $N=8$ thus becomes

$$2, 4, 6, 8, 3, 1, 7, 5.$$

Now clearly $Q_N \leqq N$. That equality may actually be achieved for every $N>3$ is proved by the following very unattractive separation into cases:

N	Solution
$4+6j$	$2, 4, \cdots, n, 1, 3, \cdots, n-1.$
$5+6j$	$2, 4, \cdots, n-1, 1, 3, \cdots, n.$
$6+6j$	$2, 4, \cdots, n, 1, 3, \cdots, n-1.$
$7+6j$	$2, 4, \cdots, n-1, 1, 3, \cdots, n.$
$8+12j$	$2, 4, \cdots, n, 3, 1, 7, 5, \cdots, n-1, n-3.$
$9+12j$	$n, 4, 6, \cdots, n-1, 3, 1, 7, 5, \cdots, n-2, n-4, 2.$
$14+12j$	$n-1, 2, 4, \cdots, n, 3, 1, 7, 5, \cdots, n-3, n-5.$
$15+12j$	$n, n-2, 2, 4, \cdots, n-1, 3, 1, \cdots, n-4, n-6.$

We conclude that $Q_N=N$.

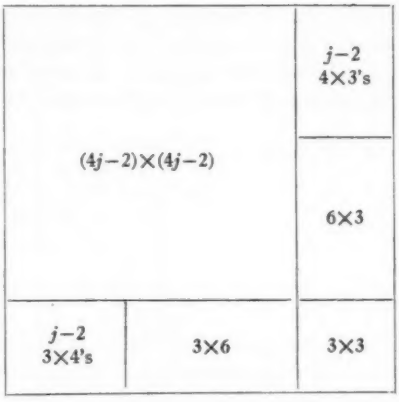
For the horse problem one soon suspects that the diagonal pattern shown earlier is best possible. The implied result would be $H_N=\frac{1}{2}N^2$, N even; $H_N=\frac{1}{2}(N^2+1)$, N odd. To show that these values cannot be exceeded a few preliminary results are helpful. For the small rectangular boards listed below a brief examination of cases suffices to establish the maxima shown.

Board	2×4	3×4	4×4	3×3	3×6	5×5	6×6
Maximum	4	6	8	5	9	13	18

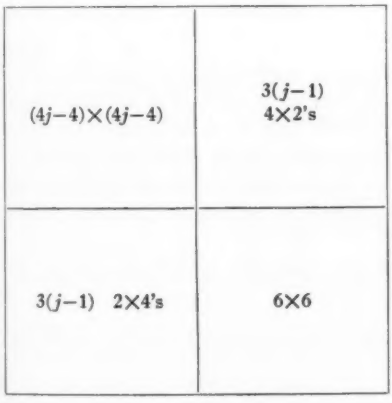
The labor involved can be reduced to a very comfortable size by partitioning the larger of these boards. For example the 6×6 may be partitioned into four 2×4's around a central 2×2. That only two of these central squares may be occupied follows quickly by trial.

The general $N\times N$ board may now be partitioned as follows:

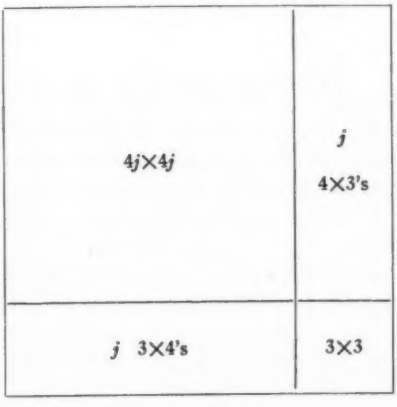
$N=4j$:
$H_N \leqq 8j^2 = \frac{1}{2}N^2$.



$N=4j+1$:
$H_n \leqq \frac{1}{2}(4j-2)^2 + 5 + 2\cdot 9 + 12(j-2) = \frac{1}{2}(N^2+1)$.



$N=4j+2$:
$H_N \leqq 8(j-1)^2+18+24(j-1) = \frac{1}{2}N^2$.



$N=4j+3$:
$H_N \leqq 8j^2+5+12j = \frac{1}{2}(N^2+1)$.

Finally, the kings. Here again one suspects that the pattern shown earlier is "optimal." The implied result is

$$K_N = \tfrac{1}{4}N^2,\ N \text{ even}; \qquad K_N = \tfrac{1}{4}(N+1)^2,\ N \text{ odd}.$$

That these values cannot be exceeded may be shown just as in the case of horses. However, the following argument, which avoids the split into cases, is interesting.

Suppose some optimal pattern is placed on the board. We will show that this

pattern can be changed to the one above by simply moving various pieces one square. First, if square A is unoccupied (Fig. 2), then exactly one of the squares B, C, D, must be occupied. (If none were occupied we could add another king at A, violating the assumption that our original pattern is optimal.) So, by mov-

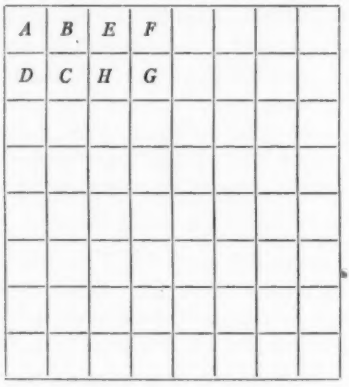


FIG. 2

ing this piece, A will be occupied and B, C, D, free. Of course, if A were occupied in the original pattern, then B, C, D, would be free and no move would be necessary. Now the same argument may be applied to E, F, G, H, leaving E occupied and G, H, F, free, and so on around the board. The result is the transformation of the original pattern, assumed optimal, to our regular pattern, which therefore must be optimal also.

It is interesting to observe that all these maximum problems are special cases of the following:

Determine the maximum of $\sum_{i=1}^{N} x_i$, *where* x_i *is either* 0 *or* 1, *subject to the constraints* $x_i \sum_{j=1}^{N} a_{ij}x_j = 0$, $i = 1, \cdots, N$.

The a_{ij} are the elements of a symmetric matrix A and are also 0 or 1; in particular, $a_{ii} = 0$. A may be termed the "challenge" matrix, each of its rows indicating with 1's the positions challenged from a particular square. For kings on a 3×3 board, the challenge matrix could be as shown in Figure 3.

We may further observe that if an optimal solution is

$$x_i = \begin{cases} 1 & \text{if } i = i_1, \cdots, i_m = \{S\}, \\ 0 & \text{otherwise,} \end{cases}$$

the constraints reduce to $\sum_{j \in \{S\}} a_{ij} = 0$, $i \in \{S\}$ and since $a_{ij} \geqq 0$, it follows that $a_{ij} = 0$ when $i \in \{S\}, j \in \{S\}$. That is, the matrix A has a zero principal minor of order M. One now sees that the problem is equivalent to finding a principal

0	1	0	1	1	0	0	0	0
1	0	1	1	1	1	0	0	0
0	1	0	0	1	1	0	0	0
1	1	0	0	1	0	1	1	0
1	1	1	1	0	1	1	1	1
0	1	1	0	1	0	0	1	1
0	0	0	1	1	0	0	1	0
0	0	0	1	1	1	1	0	1
0	0	0	0	1	1	0	1	0

FIG. 3

minor of A having maximum order and consisting only of zeros. In the example above, this maximum order is, of course, four, and the appropriate elements have been underscored (Fig. 3). In general, no useful scheme for discovering this minor exists.

A CLASS OF GROUPS ASSOCIATED WITH LATIN SQUARES*

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Let (a_{ij}) be the $n \times n$ Latin square whose rows are the successive cyclic permutations of the integers 0, 1, $\cdots$, $n-1$, so that $a_{ij} \equiv i+j \pmod{n}$. Let

$$(x_i): x_0, x_1, \cdots, x_{n-1},$$

be an arbitrary permutation of the integers 0, 1, $\cdots$, $n-1$. If the rows (columns) are numbered 0, 1, $\cdots$, $n-1$ from top to bottom (left to right), the element x_i will be found in column i and row r_i, $i=0, 1, \cdots, n-1$, where $r_i \equiv x_i - i \pmod{n}$. The permutation (x_i) will be called a 1-*permutation* if $r_i \neq r_j$ for $i \neq j$, that is, if the integers $r_0, r_1, \cdots, r_{n-1}$, also form a permutation of the integers 0, 1, $\cdots$, $n-1$. We seek, for a given n, the number $N(n)$ of 1-permutations. $N(n)$ is the number of ways n distinct integers can be selected from the Latin square (a_{ij}) so that no two integers are in the same row or in the same column.

If n is even, $N(n)=0$. To see this, let (x_i) be an arbitrary permutation and let r_i be defined as above. Then

$$\sum_{i=0}^{n-1} r_i \equiv \sum_{i=0}^{n-1} (x_i - i) \equiv \sum_{i=0}^{n-1} x_i - \sum_{i=0}^{n-1} i \equiv 0 \pmod{n}.$$

* Presented to the International Congress of Mathematicians 1958, Edinburgh, Scotland on August 16, 1958.

Also $\sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1)$ and is therefore congruent to 0 (mod n) if and only if n is odd. Hence the r's, which are integers in the range 0, 1, $\cdots$, $n-1$, cannot all be distinct if n is even.

The function $N(n)$ is unknown at present for n odd but the following values have been calculated: $N(1)=1$, $N(3)=3$, $N(5)=15$, $N(7)=133$, $N(9)=2025$, $N(11)=37{,}851$. In an effort to shed some light on the values of $N(n)$ for larger values of n, we investigate a certain class of groups.

We first determine, for each positive integer n, an abstract group $\mathfrak{G}$ of order $6n^2\phi(n)$, where $\phi(n)$ is the familiar Euler ϕ-function, and then relate $\mathfrak{G}$ to the problem at hand. $\mathfrak{G}$ is defined as the product of four of its subgroups, $\mathfrak{G}_{12}$, $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{K}$, as follows. $\mathfrak{G}_{12}$ is the group of order 12 generated by two elements C and R with the generating relations

$$C^2 = R^2 = (CR)^6 = I, \tag{1}$$

where I is the identity of the group.

If we put

$$B = CR, \quad D = CB^2, \quad E = CB^3, \quad F = CB^4, \quad G = CB^5,$$

it is easy to verify that $\mathfrak{G}_{12}$ is indeed of order 12 and has the multiplication table

I	C	R	D	E	F	G	B	B^2	B^3	B^4	B^5
C	I	B	B^2	B^3	B^4	B^5	R	D	E	F	G
R	B^5	I	B	B^2	B^3	B^4	D	E	F	G	C
D	B^4	B^5	I	B	B^2	B^3	E	F	G	C	R
E	B^3	B^4	B^5	I	B	B^2	F	G	C	R	D
F	B^2	B^3	B^4	B^5	I	B	G	C	R	D	E
G	B	B^2	B^3	B^4	B^5	I	C	R	D	E	F
B	G	C	R	D	E	F	B^2	B^3	B^4	B^5	I
B^2	F	G	C	R	D	E	B^3	B^4	B^5	I	B
B^3	E	F	G	C	R	D	B^4	B^5	I	B	B^2
B^4	D	E	F	G	C	R	B^5	I	B	B^2	B^3
B^5	R	D	E	F	G	C	I	B	B^2	B^3	B^4

We note that the group $\mathfrak{G}_{12}$ is independent of the value of n.

Let $m_1, m_2, \cdots, m_{\phi(n)}$ be the positive integers not greater than and prime to n. They form a group $\mathfrak{M}^*$ under multiplication, modulo n, of order $\phi(n)$. We define a group $\mathfrak{M}$, with elements

$$M_{m_1}, M_{m_2}, \cdots, M_{m_{\phi(n)}},$$

to be an abstract group isomorphic to $\mathfrak{M}^*$.

Finally, $\mathfrak{A}$ and $\mathfrak{K}$ are defined to be cyclic groups, each of order n, with generators A and K, respectively. M_{n-1} is an element of $\mathfrak{M}$ if $n>1$ since then $n-1$ is a positive integer prime to n. Let m be an arbitrary element of $\mathfrak{M}^*$ and m', its inverse (so that $mm' \equiv 1 \pmod{n}$); then $M_m M_{m'} = I$ in $\mathfrak{M}$. We now stipulate, in addition to (1), the following generating relations.

$$(CR)^2 = M_{n-1}; \tag{2}$$

$$M_m C = CM_m, \qquad M_m R = RM_m; \tag{3}$$

$$AK = KA; \tag{4}$$

$$AC = CA^{n-1}, \qquad AR = RK^{n-1}, \qquad AM_m = M_m A^m, \tag{5}$$

$$KC = CA^{n-1}K, \qquad KR = RA^{n-1}, \qquad KM_m = M_m K^m. \tag{6}$$

If W, X, Y, Z, are arbitrary elements of $\mathfrak{S}_{12}$, $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{K}$, respectively, it follows, by use of the relations (3), (4), (5), (6), that the product of two elements of the form $WXYZ$ is again an element of this form. There are $12n^2\phi(n)$ products of this form, but because of the relation (2), these fall into half as many distinct pairs of equal elements. We thus have

THEOREM 1. *The elements C, R, M_m, A, K, defined as above, and subject to the generating relations* (1) *through* (6), *generate a group $\mathfrak{G}$ of order $6n^2\phi(n)$.*

To relate the group $\mathfrak{G}$ to 1-permutations and the evaluation of $N(n)$, we will find it convenient to introduce a new symbol for a permutation (x_i) of the integers $0, 1, \cdots, n-1$. Let (y_i) also be a permutation of the integers $0, 1, \cdots, n-1$; we write the symbol $(x_i | y_i)$ to designate the permutation which has the integer x_i in the y_ith place (place numbers are read from left to right), $i=0, 1, \cdots, n-1$. Thus, the permutation (x_i) can now be written as $(x_i | i)$; the permutation obtained by reversing the order of the elements of (x_i) can be written as $(x_i | n-1-i)$ or $(x_{n-1-i} | i)$. If $(x_i | y_i)$ is a permutation, the permutation obtained by moving each element two spaces to the left (the 0th element will go into the $(n-2)$nd place, the first element into the $(n-1)$st place) is $(x_i | y_i - 2)$. It is understood, of course, that any integer which is an element of a permutation or the place number of an element or a subscript on either can be replaced by any other integer congruent to it (mod n).

THEOREM 2. *If (x_i) and (y_i) are permutations of the integers* $0, 1, \cdots, n-1$, *the permutation $(x_i | y_i)$ is a 1-permutation if and only if $(x_i - y_i)$, $i=0, 1, \cdots, n-1$, is a permutation of the integers* $0, 1, \cdots, n-1$.

The proof of the theorem follows immediately from the observation that, for a given i, the element x_i of the permutation $(x_i|y_i)$ is found in row x_i-y_i and column y_i of the Latin square (a_{ij}).

As an immediate consequence of this theorem, we have

THEOREM 3. *If the permutation $(x_i|y_i)$ is a 1-permutation, so are the permutations $(y_i-x_i|y_i)$, $(y_i|x_i)$, $(mx_i|my_i)$, $(x_i+a|y_i)$, $(x_i|y_i+a)$, where m is an integer relatively prime to n and a is any integer.*

The 1-permutations $(y_i-x_i|y_i)$ and $(y_i|x_i)$ are called the *complement* and the *reverse*, respectively, of the 1-permutation $(x_i|y_i)$; the 1-permutations $(mx_i|my_i)$, $(x_i+a|y_i)$, $(x_i|y_i+a)$ are said to be *multiplicatively*, *additively*, and *cyclically equivalent*, respectively, to the 1-permutation $(x_i|y_i)$.

We now consider a number of operators that transform 1-permutations into 1-permutations. In particular, we define operators $\overline{I}$, $\overline{C}$, $\overline{R}$, $\overline{M}_m$, $\overline{A}$, $\overline{K}$, where n is an odd positive integer and m is an integer not greater than and prime to n, by the statements

$$(7)\quad \begin{aligned} &\overline{I}(x_i|y_i) = (x_i|y_i), \quad \overline{C}(x_i|y_i) = (y_i - x_i|y_i), \quad \overline{R}(x_i|y_i) = (y_i|x_i), \\ &\overline{M}_m(x_i|y_i) = (mx_i|my_i), \quad \overline{A}(x_i|y_i) = (x_i+1|y_i), \quad \overline{K}(x_i|y_i) = (x_i|y_i-1). \end{aligned}$$

Products of the operators are defined as usual; for example,

$$\begin{aligned} \overline{C}^2(x_i|y_i) &= \overline{C}[\overline{C}(x_i|y_i)] = \overline{C}(y_i - x_i|y_i) = (x_i|y_i) = \overline{I}(x_i|y_i), \\ \overline{C}\overline{R}(x_i|y_i) &= \overline{C}[\overline{R}(x_i|y_i)] = \overline{C}(y_i|x_i) = (x_i - y_i|x_i). \end{aligned}$$

It is readily seen, by direct computation and straightforward use of the definitions, that the operators defined above satisfy relations identical to the relations (1) through (6). Hence, corresponding to any element $WXYZ$, defined as above, of the group $\mathfrak{G}$ there is an operator $\overline{W}\overline{X}\overline{Y}\overline{Z}$ that transforms a 1-permutation into a 1-permutation. Furthermore, if $W_1X_1Y_1Z_1 \cdot W_2X_2Y_2Z_2 = W_3X_3Y_3Z_3$, then $\overline{W}_1\overline{X}_1\overline{Y}_1\overline{Z}_1 \cdot \overline{W}_2\overline{X}_2\overline{Y}_2\overline{Z}_2 = \overline{W}_3\overline{X}_3\overline{Y}_3\overline{Z}_3$.

We observe, however, that if $T_1 = W_1X_1Y_1Z_1$ and $T_2 = W_2X_2Y_2Z_2$ are distinct elements of $\mathfrak{G}$, the corresponding operators $\overline{T}_1 = \overline{W}_1\overline{X}_1\overline{Y}_1\overline{Z}_1$ and $\overline{T}_2 = \overline{W}_2\overline{X}_2\overline{Y}_2\overline{Z}_2$ may be identical for a particular 1-permutation; that is, the two operators, although they correspond to distinct elements of $\mathfrak{G}$, may both transform a given 1-permutation into the same 1-permutation. It is to be emphasized that whether the operators $\overline{T}_1$ and $\overline{T}_2$ are or are not identical depends on the 1-permutation upon which they act. Thus, 0, 2, 1 and 1, 0, 2 are two of the three 1-permutations for $n=3$; we have

$$\overline{C}(0, 2, 1) = \overline{I}(0, 2, 1) = (0, 2, 1),$$

where $\overline{C}(0, 2, 1)$ is an abbreviation for $\overline{C}(0, 2, 1|0, 1, 2)$, etc. On the other hand,

$$\overline{C}(1, 0, 2) = (2, 1, 0) \neq \overline{I}(1, 0, 2) = (1, 0, 2).$$

Let p be a specific 1-permutation for some fixed n and let $\mathfrak{K}(p)$ be the totality of elements of $\mathfrak{G}$ whose corresponding operators leave p unaltered. Then $\mathfrak{K}(p)$ is a subgroup of $\mathfrak{G}$ of order $h(p)$, say. The subgroup $\mathfrak{K}(p)$ determines a partitioning of $\mathfrak{G}$ into $t=6n^2\phi(n)/h(p)$ cosets, each of the form $g\mathfrak{K}(p)$. The $h(p)$ operators corresponding to the $h(p)$ elements of $\mathfrak{G}$ in one coset will all transform p into the same 1-permutation q and this set of $h(p)$ operators will contain all the operators that transform p into q. It follows that the operators will transform p into t distinct 1-permutations, say, $p=p_1, \cdots, p_t$. Let $\overline{Q}_1, \cdots, \overline{Q}_t$ be a set of operators such that $\overline{Q}_i$ transforms $p=p_1$ into p_i, $i=1, \cdots, t$. The $\overline{Q}$'s form a group $\bar{\mathfrak{Q}}(p)$ of order t; it follows that $\mathfrak{K}(p)$ is an invariant subgroup of $\mathfrak{G}$ and $\bar{\mathfrak{Q}}(p)$ is (simply) isomorphic to the factor group $\mathfrak{G}/\mathfrak{K}(p)$.

The 1-permutations $p_1, \cdots, p_t$ form a transitive system; the various operators merely permute these p's among themselves and no operator, corresponding to an element of $\mathfrak{G}$, will transform a 1-permutation not in this set into a 1-permutation of this set.

We can continue in the same manner and separate the $N(n)$ 1-permutations into a certain number of disjoint transitive systems such that the operators corresponding to the elements of $\mathfrak{G}$ merely permute the 1-permutations of a transitive system among themselves.

These ideas can be pushed a bit further. Let p and q be distinct 1-permutations (for the same n). As above, they determine invariant subgroups $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ of $\mathfrak{G}$. If p and q belong to the same transitive system, $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ will be of the same order; if p and q belong to different transitive systems, $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ may be of the same or of different orders. In either case, $\mathfrak{K}(p, q) = \mathfrak{K}(p) \cap \mathfrak{K}(q)$ will be an invariant subgroup of $\mathfrak{G}$ such that the operators corresponding to its elements will leave both p and q individually unaltered. More generally, if $p, q, r, \cdots, u$ is any set of 1-permutations, then

$$\mathfrak{K}(p, q, \cdots, u) = \mathfrak{K}(p) \cap \mathfrak{K}(q) \cap \cdots \cap \mathfrak{K}(u)$$

will be an invariant subgroup of $\mathfrak{G}$ whose associated operators will leave $p, q, \cdots, u$ individually unaltered. In particular, if $p, q, \cdots, v$ is the set of all $N(n)$ 1-permutations, then $\mathfrak{K}=\mathfrak{K}(p, q, \cdots, v)=\mathfrak{K}(p)\cap\mathfrak{K}(q)\cap\cdots\cap\mathfrak{K}(v)$ will be an invariant subgroup of $\mathfrak{G}$ whose associated operators will leave each of the $N(n)$ 1-permutations individually unaltered.

If $n=3$, $N(3)=3$; let p, q, r be the three distinct 1-permutations. The group $\mathfrak{G}$ is of order 108; the subgroups $\mathfrak{K}(p), \mathfrak{K}(q), \mathfrak{K}(r)$ are distinct but each is of order 36 so that p, q and r form a single transitive system; the subgroup $\mathfrak{K}$ is of order 18. This means that in addition to the identity operator, there are 5 operators that actually alter at least one of the three 1-permutations.

If $n=5$, $N(5)=15$, and $\mathfrak{G}$ is of order 600. The 15 1-permutations form a single transitive system; if p is any one of the 15 1-permutations, $\mathfrak{K}(p)$ is a subgroup of order 40. The subgroup $\mathfrak{K}$ reduces to the identity of $\mathfrak{G}$.

It appears that for $n>3$, $\mathfrak{K}$ always reduces to the identity of $\mathfrak{G}$.

As a final example, we take $n=11$, then $\mathfrak{G}$ is of order 7260 and $N(11)=37{,}851$. These 1-permutations are divided into 11 disjoint transitive sets. We list a representative 1-permutation from each set and give the total number of distinct transforms of the representative 1-permutation.

1-permutations											Number of distinct transforms
0	2	4	6	8	10	1	3	5	7	9	33
0	3	6	9	1	4	7	10	2	5	8	66
0	2	4	8	3	9	1	10	5	7	6	726
0	2	5	8	3	1	10	9	6	4	7	726
0	2	4	6	9	1	10	5	3	8	7	3630
0	2	4	7	9	1	5	10	3	6	8	3630
0	2	5	8	10	9	3	6	4	7	1	3630
0	2	5	9	6	10	4	3	1	8	7	3630
0	2	4	7	1	10	9	6	3	5	8	7260
0	2	4	7	1	10	9	5	3	8	6	7260
0	2	4	7	1	10	9	3	6	8	5	7260
											37851

The group $\mathfrak{G}$ provides some additional information about the 1-permutations and $N(n)$. Each operator corresponding to an element of the subgroup $\mathfrak{G}_{12}\mathfrak{M}$ of $\mathfrak{G}$ transforms a 1-permutation beginning with the integer 0 into a 1-permutation also beginning with the integer 0 (although an operator with this property is not necessarily an operator corresponding to an element of $\mathfrak{G}_{12}\mathfrak{M}$).

The operator $\overline{C}$ transforms a 1-permutation beginning with the integers 0, x into a 1-permutation beginning with the integers 0, $1-x$. Hence the number of 1-permutations beginning with 0, x is equal to the number beginning with 0, $1-x$.

Similarly, if $n=2k-1$, then $(k, n)=1$ and $2k\equiv 1 \pmod{n}$. The operator $\overline{M}_k\overline{R}$ transforms a 1-permutation beginning with 0, 2 into a 1-permutation beginning with 0, k. Hence the numbers of 1-permutations beginning with 0, 2; 0, k; 0, $n-1$ are equal.

The operator $\overline{A}$ transforms a 1-permutation beginning with the integer x into a 1-permutation beginning with the integer $x+1$; hence the number of 1-permutations beginning with one integer is the same as the number of 1-permutations beginning with any other integer (in the range $0, 1, \cdots, n-1$) It follows that $N(n)\equiv 0 \pmod{n}$.

FOUR INTERSECTING SPHERES

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Summary. Four intersecting spheres (A), (B), (C), (D), taken three at a time determine four pairs of points (e).

To study the configuration formed by those points we consider: (a) The sphere (M) orthogonal to the given spheres; (b) the tetrahedron $(T)=ABCD$ formed by the centers of those spheres; (c) the polar reciprocal tetrahedron $(T')=A'B'C'D'$ of (T) with respect to (M); (d) the "adjoint spheres" (A'), (B'), (C'), (D'), with centers at the vertices of (T') and orthogonal to (M).

The eight points (e) give rise to $2^4=16$ "tetrahedrons of intersection" of the given spheres. They may be grouped into eight "pairs of complementary tetrahedrons of intersection" (abbreviated to c.t.i.), each pair involving all the eight points.

The tetrahedrons of each pair are perspective, the center of perspectivity being the center M of the sphere (M). Their eight planes of perspectivity coincide with the eight planes of similitude of the four adjoint spheres.

The radical plane of the circumspheres of a pair of c.t.i. coincides with the plane of perspectivity of the two tetrahedrons considered. The two spheres are inverse with respect to (M), are cut isogonally by each of the given spheres, etc.

1. Preliminaries. a. Given four spheres (A), (B), (C), (D), whose centers form a tetrahedron $(T)=ABCD$, the "central tetrahedron" of the given spheres, let

$$(e) \qquad P,\ P';\ Q,\ Q';\ R,\ R';\ S,\ S'$$

be, respectively, the pairs of points of intersection of the triads of spheres

$$(g) \qquad (B),\ (C),\ (D);\ (C),\ (D),\ (A);\ (D),\ (A),\ (B);\ (A),\ (B),\ (C).$$

The points of intersection (e) located on each sphere are listed in

TABLE 1

(A)			$Q,\ Q';$		$R,\ R';$		$S,\ S'$
(B)	$P,\ P';$				$R,\ R';$		$S,\ S'$
(C)	$P,\ P';$		$Q,\ Q';$				$S,\ S'$
(D)	$P,\ P';$		$Q,\ Q';$		$R,\ R'$		

b.i. Two points of the same pair of (e) will be referred to as "complementary points of intersection" of the four given spheres.

ii. A line joining two noncomplementary points of intersection will be said to be a "line of intersection" of the given spheres. Their two complementary points determine the "complementary line of intersection." Two pairs of complementary points of intersection, say, $P,\ P';\ Q,\ Q'$ determine two pairs of

complementary lines of intersection: PQ, $P'Q'$; PQ', $P'Q$.

iii. Three points of intersection, no two of them complementary, determine a "plane of intersection." The three complementary points determine the "complementary plane of intersection" of the given spheres.

iv. Four points of intersection, no two of them complementary, form a "tetrahedron of intersection" of the four given spheres. The four remaining points of intersection form the "complementary tetrahedron of intersection"; this locution is abbreviated to c.t.i. in what follows, to save space.

2. Perspective tetrahedrons. a. The lines PP', QQ', RR', SS', are the radical axes of the triads of spheres (g), respectively (Sec. 1a), hence these four lines meet in the orthogonal center M of the four given spheres, which point is also the center of the orthogonal sphere (M) of those spheres.

If we take one point, and only one, in each of the four pairs of points (e) (Sec. 1a) we obtain $2^4=16$ tetrahedrons of intersection which may be grouped into eight pairs of c.t.i. (Sec. 1b iv). Two such tetrahedrons involve all the eight points (e).

b. THEOREM. *A pair of c.t.i. are perspective.*

Indeed, pairs of corresponding vertices of two such tetrahedrons are alined on the orthogonal center M of the given spheres (Sec. 2a). The eight pairs of c.t.i. have thus the same center of perspectivity. Their eight planes of perspectivity will be discussed later.

3. The radical tetrahedron. a. The polar planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ of the orthocenter M (Sec. 2a) for the spheres (A), (B), (C), (D), respectively, form a tetrahedron $(T')=A'B'C'D'$ which, for want of a better term, may be referred to as the "radical tetrahedron" of the four given spheres. The following properties of the radical tetrahedron (T') are mentioned here, for future reference.

b. The sphere (M) being orthogonal to the four spheres (A), (B), (C), (D), the faces of (T') are, respectively, the radical planes of the four pairs of spheres (M), (A); (M), (B); (M), (C); (M), (D).

c. The radical plane $B'C'D'$ of the two orthogonal spheres (M), (A) (Sec. 3b) is also the polar plane of the center A of (A) with respect to (M). Similarly for the other faces of (T'). Thus: The central tetrahedron $(T)=ABCD$ (Sec. 1a) and the radical tetrahedron (T') of the four given spheres are polar reciprocal with respect to the orthogonal sphere (M) of the given spheres.

d. The vertex A' of (T') is the pole of the face BCD of (T) for (M) (Sec. 3c), hence the perpendicular from M upon the plane BCD passes through A'. Now that perpendicular is the radical axis PP' of the three spheres (B), (C), (D) (Sec. 2a), hence: The vertices of the radical tetrahedron of four given spheres lie

on the four radical axes of those spheres taken three at a time.

We have thus the tetrads of collinear points:

$$(j) \qquad MMP'A',\ MQQ'B',\ MRR'C',\ MSS'D'.$$

4. Harmonic relations. a. Consider any two of the four concurrent lines (j) (Sec. 3d), say, $MQQ'B'$, $MRR'C'$. Their plane cuts the sphere (D) along a small circle (d) (Fig. 1) passing through the points Q, Q', R, R', since these four points lie both in the plane considered and on the sphere (D) (Table 1).

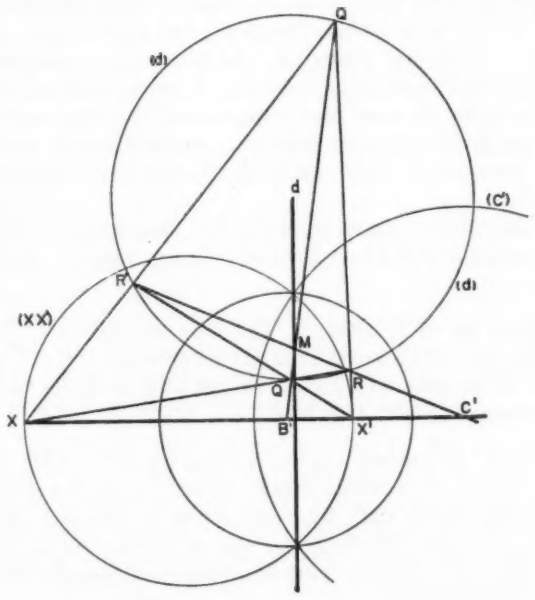

FIG. 1

The diagonal triangle of the complete quadrangle $QQ'RR'$ inscribed in (d) has for its vertices the points M, $X=(QR, Q'R')$, $X'=(QR', Q'R)$; the diagonal triangle MXX' being polar to (d), the line XX' is the polar of M for that circle.

On the other hand, the polar plane $A'B'C'$ of M for the sphere (D) (Sec. 3a) cuts the plane $(MQQ'B', MRR'C')$ of the circle (d) along the line $B'C'$, and this line is the polar of M for (d) ([1], p. 141, art. 426), hence the line $B'C'$ coincides with XX', and we have $((j)$, Sec. 3d$)$: $B'=(MQQ', XX')$, $C'=(MRR'C', XX')$.

Thus the points B', C' are harmonically separated from the point M by two pairs of points Q, Q'; R, R', respectively.

Considering the pair of lines $MPP'A'$, $MSS'D'$ of (j) we obtain similar results for the points A', D'. We have thus the four groups of *harmonic* pairs of points:

(*h*) MA', PP'; MB', QQ'; MC', RR'; MD', SS',

located on four concurrent lines.

b. The points X, X' (Sec. 4a) are determined by the two pairs of complementary lines of intersection QR, $Q'R'$; QR', $Q'R$ derived from the two pairs of complementary points of intersection Q, Q'; R, R'. Moreover, X, X' lie on the edge $B'C'$ of (T') joining the vertices B', C' of (T') (Sec. 4a) situated on the lines MQQ', MRR'.

Similarly for any other pair of the four lines (*j*) (Sec. 3d). Hence: i. *A pair of complementary lines of intersection of the four given spheres meet on an edge of the radical tetrahedron of the four spheres*; ii. *The other pair of complementary lines of intersection determined by the same two pairs of complementary points of intersection* (Sec. 16ii) *meet on the same edge of the radical tetrahedron as the first pair of lines.* iii. *The two points of intersection thus obtained on the edge of the radical tetrahedron are harmonically separated by the two vertices of that tetrahedron located on that edge.*

Indeed, in the complete quadrangle $QQ'RR'$ (Sec. 4a) the two vertices X, X' of the diagonal triangle MXX' are harmonically separated by the sides $MQQ'B'$, $MRR'C'$.

5. The adjoint set of spheres. a. The four spheres (A'), (B'), (C'), (D'), having for centers the vertices of the radical tetrahedron $(T') = A'B'C'D'$ (Sec. 3) and orthogonal to the sphere (M) (Sec. 2a) are said to be the "adjoint spheres" of the given spheres (A), (B), (C), (D).

b. The point A' is common to the radical planes of (M) with the three spheres (B), (C), (D), respectively (Secs. 3b, 3d), hence the sphere (A') which is orthogonal to the sphere (M), by construction, is also orthogonal to each of the spheres (B), (C), (D). Similarly for the other vertices of (T'). Thus: *The adjoint spheres* (A'), (B'), (C'), (D') *are respectively orthogonal to the corresponding triads of spheres* (*g*) (Sec. 1a).

c. The center A' of the sphere (A') orthogonal to the spheres (B), (C), (D) lies on their radical axis which is the line PP' (Sec. 1a). We have thus another proof of a property noted already (Sec. 3d).

6. Centers of similitude. a. The two spheres (B'), (C') are orthogonal to the sphere (D) (Sec. 5b), and their centers lie in the plane (MQQ', MRR') of (d) (Sec. 5c), hence this plane cuts the two spheres (B'), (C') along two great circles (b'), (c') orthogonal to the circle (d) ([1], p. 176, ex. 19a). (Fig. 1).

The two diagonal points X, X' (Sec. 4a) are conjugate with respect to the circle (d) (Sec. 4a), hence the circle (XX') having XX' for diameter is orthogonal to (d) ([2], p. 181, art. 387).

Furthermore, the points X, X' are harmonically separated by the centers B', C' of the circles (b'), (c') (Sec. 4biii), therefore (XX') is the circle of simili-

tude of the circles (b'), (c') ([2], p. 214, art. 481; [3]). Hence the points X, X' are the two centers of similitude of the two great circles (b'), (c') [5], and therefore also of the two spheres (B'), (C'). Consequently: *Given four intersecting spheres, the two pairs of complementary lines of intersection determined by two pairs of complementary points of intersection* (Sec. 1aii) *meet in two points on an edge of the radical tetrahedron of the given spheres. The two points thus obtained are the two centers of similitude of the two adjoint spheres having their centers on the edge considered.*

7. Axes of similitude. a. Consider any pair of complementary planes of intersection (Sec. 1biii), say, PQR, $P'Q'R'$.

The pair of complementary lines of intersection QR, $Q'R'$ intersect in a point X on the edge $B'C'$ of (T') (Sec. 6), and X is a center of similitude of the spheres (B'), (C'). Similarly the points $Y=(RP, R'P')$, $Z=(PQ, P'Q')$ lie, respectively, on the edges $C'A'$, $A'B'$, of (T'), and Y, Z are centers of similitude of the pairs of spheres (C'), (A'); (A'), (B'), respectively.

On the other hand, the point $X=(QR, Q'R')$ lies both in the plane PQR and in the plane $P'Q'R'$, for the lines QR, $Q'R'$ lie in the planes PQR, $P'Q'R'$, respectively; similarly for the points Y and Z. Hence the three points X, Y, Z lie on the straight line common to the three planes PQR, $P'Q'R'$, $A'B'C'$, and the line XYZ is an axis of similitude of the three spheres (A'), (B'), (C'). Consequently: *A pair of complementary planes of intersection of four intersecting spheres meet in a face of the radical tetrahedron of the given spheres, and this common line is an axis of similitude of three adjoint spheres.*

b. Consider the plane $XYZ-M$ determined by the axis XYZ (Sec. 7a) and the orthogonal center M. The harmonic conjugate of this plane with respect to the pair of complementary planes of intersection PQR, $P'Q'R'$ passing through XYZ meets the lines MPP', MQQ', MRR', in the harmonic conjugates of M with respect to the pairs of points P, P'; Q, Q'; R, R', that is, in the points A', B', C', respectively ((*h*), Sec. 4a). Thus the axis of similitude common to the planes PQR, $P'Q'R'$ lies in the face $A'B'C'$ of the tetrahedron (T').

Similarly for any other pair of complementary planes of intersection. Thus: *Two complementary planes of intersection are coaxal with that face of the radical tetrahedron* (T') *which contains the vertices of* (T') *lying on the lines joining the three pairs of complementary points of intersection* ((*e*), Sec. 1) *which determine the pair of planes considered.*

8. Planes of similitude. a. Consider now any two c.t.i., (Sec. 1 biv) say, $PQRS$ and $P'Q'R'S'$. Their faces are four pairs of complementary planes of intersection, and therefore meet in four lines which are four axes of similitude of the adjoint spheres (Sec. 7a).

On the other hand, the two tetrahedrons considered are perspective (Sec. 2a), hence the four axes of similitude lie in the plane of perspectivity of the two tetrahedrons. Thus ([1], pp. 157 ff.): *The plane of perspectivity of two c.t.i. of*

four intersecting spheres is a plane of similitude of the adjoint spheres of the four given spheres.

b. The four axes of similitude determined by the four pairs of corresponding faces of two c.t.i., considered lie in four different faces of the radical tetrahecron (T'). The face of (T') containing the line of intersection of a given pair of corresponding faces of any two c.t.i. is readily determined. For instance, if the two c.t.i. $PQ'RS'$, $P'QR'S$ be considered, their pair of corresponding faces PRS', $P'R'S$ intersect in the face $A'C'D'$ of (T'), by Section 7b.

The preceding proposition (Sec. 8a) may be stated in other words: i. *The eight planes of perspectivity of the eight pairs of c.t.i. of four given intersecting spheres coincide with the eight planes of similitude of the four adjoint spheres of the given spheres.* ii. *The eight planes of perspectivity may be grouped into two tetrahedrons which are mutually harmonic and which form a desmic system with the radical tetrahedron of the four given spheres* ([1], p. 240, art. 736).

d. It may also be noticed that: The radical tetrahedron (T') is perspective to each tetrahedron of intersection, the center of perspectivity being the orthogonal center of the given spheres ((*j*), Sec. 3d and (*h*), Sec. 4a), the planes of perspectivity coinciding with the planes of similitude of the adjoint spheres (Secs. 8a, 8c).

9. Spheres of intersection. a. The circumsphere of a tetrahedron of intersection is, by definition, a "sphere of intersection" of the four given spheres. The circumspheres of two c.t.i are two "complementary spheres of intersection."

b. Consider two complementary spheres of intersection, say, $(O)=PQRS$, $(O')=P'Q'R'S'$. The two points P, P' are collinear with the center M of the sphere (M) and lie on the sphere (B) orthogonal to (M), hence P, P' are inverse for (M). Similarly for the other pairs of points (*e*) (Sec. 1a). Hence the spheres (O), (O') are inverse with respect to (M) as sphere of inversion. Consequently ([1], pp. 218, 219; arts. 672, 673): *Two complementary spheres of intersection of four given spheres are coaxal with the orthogonal sphere of the latter, and have that sphere as one of their spheres of antisimilitude.*

c. The two points P, P' of the spheres (O), (O') being inverse for the sphere (M), are antihomologous on the spheres (O), (O'). Similarly for the other pairs of points (*e*). Hence any two corresponding faces of the two perspective tetrahedrons $PQRS$, $P'Q'R'S'$ (Sec. 2), say, PQR, $P'Q'R'$ are antihomologous planes for the spheres (O), (O') and therefore intersect in the radical plane of the two spheres. Thus the radical plane of the spheres (O), (O') coincides with the plane of perspectivity of the tetrahedrons $PQRS$, $P'Q'R'S'$.

The same argument holds for any other pair of c.t.i. and their respective circumspheres. Hence: *The eight radical planes of the eight pairs of complementary spheres of intersection of four given intersecting spheres coincide with the eight planes of similitude of the group of adjoint spheres of the given spheres* (Sec. 8c).

10. Isogonal spheres. a. The spheres (O), (O') being coaxal with the sphere (M) (Sec. 9b), their centers O, O' are collinear with M, and the line of centers MOO' is perpendicular to the radical plane of (O), (O'). Similarly for any other pair of complementary spheres of intersection of the four given spheres. Thus (Sec. 9c): *Given four intersecting spheres, the perpendicular dropped from their orthogonal center upon the plane of perspectivity of a pair of their c.t.i. passes through the circumcenters of those two tetrahedrons.*

b. The given sphere (A) cuts the two spheres $(O)=PQRS$, $(O')=P'Q'R'S'$ along the circles QRS, $Q'R'S'$, respectively (see Sec. 1, Table 1, line 1), and (A) is orthogonal to the sphere of antisimilitude (M) of (O), (O') (Sec. 9b); hence (A) cuts the two spheres (O), (O') isogonally ([1], p. 223, art. 685). The same sphere (A) cuts isogonally any other pair of complementary spheres of intersection of the four given spheres, for analogous reasons.

The same argument applied to the sphere (A) is valid for any other of the four given spheres (Sec. 1). Consequently: *Given four intersecting spheres, each of them cuts isogonally any complementary pair of their spheres of intersection.*

11. Spheres of antisimilitude. a. The two adjoint spheres (A'), (B') are coaxal with their two spheres of antisimilitude ([1], p. 185), and the sphere (M) being orthogonal to the former two spheres (Sec. 5a), is also orthogonal to the latter two ([1], p. 179, art. 566). Thus the orthogonal sphere (M) of the four spheres (A'), (B'), (C'), (D'), (Sec. 5a) is orthogonal to their six pairs of spheres of antisimilitude.

A plane of similitude, say, σ of the four adjoint spheres contains six centers of similitude, and the six corresponding spheres of antisimilitude form a coaxal net (N) ([1], p. 196, art. 616), the conjugate pencil of which includes the sphere (M) and has for its axis the perpendicular k from M upon σ. Now the plane σ is the radical plane of a pair of complementary spheres of intersection, say, (I), (I') (Sec. 9c), and if (U) is a sphere of antisimilitude having its center in σ, this sphere (U) is orthogonal to (M) and therefore also to the spheres (I), (I') coaxal with (M) (Sec. 9b). Consequently: *Given four intersecting spheres, any pair of complementary spheres of intersection is orthogonal to six spheres of antisimilitude of the adjoint group of spheres. The six centers of those six spheres lie in the plane of similitude which coincides with the radical plane of the two spheres of intersection.*

The reader may consider, in this connection, the spheres of similitude of the adjoint spheres considered.

b. Each pair of complementary spheres of intersection being orthogonal to six spheres of antisimilitude (Sec. 10a), the eight pairs of such spheres are orthogonal to $8\times 6=48$ spheres of antisimilitude. Now the four adjoint spheres have only twelve spheres of antisimilitude, hence each of them is orthogonal to four pairs of complementary spheres of intersection. Thus: *A sphere of antisimiltitude of the adjoint spheres of four given intersecting spheres is orthogonal to*

eight spheres of intersection of the given spheres.

12. A reciprocity. We considered two sets of spheres (A), (B), (C), (D); (A'), (B'), (C'), (D'), having a common orthogonal sphere (M), the tetrahedrons $(T)=ABCD$, $(T')=A'B'C'D'$ formed by their centers being polar reciprocal with respect to (M).

We started with the first set of spheres (A), $\cdots$, the sphere (M), and the tetrahedron (T); from this set of elements we derived the spheres (A'), $\cdots$, and (T'). The two tetrahedrons (T), (T') being polar reciprocal for (M), if we started with the second set of spheres (A'), $\cdots$, and the sphere (M), we would, by the same construction, obtain the first set (A), $\cdots$; the two tetrahedrons would interchange roles: (T') would be the central tetrahedron and (T) the radical tetrahedron.

13. Special cases. In the preceding considerations it has been tacitly assumed that no two of the four given spheres (A), $\cdots$ are orthogonal. If those four spheres are all mutually orthogonal, the central tetrahedron (T) is orthocentric and coincides with its polar reciprocal tetrahedron (T') for the sphere (M). This case has been considered elsewhere [4].

A number of intermediate cases may be considered, like (the symbol $\perp$ stands for "orthogonal"): i. $(A)\perp(B)$, $(C)\perp(D)$; ii. $(A)\perp(B)$, $(A)\perp(C)$, $(A)\perp(D)$; iii. $(A)\perp(C)$, $(A)\perp(D)$, $(B)\perp(C)$, $(B)\perp(D)$.

For lack of space, if for no other reason, these considerations are left for the entertainment of the interested reader.

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MATHEMATICAL NOTES

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VOLUMES AND AREAS OF CROSS-SECTIONS

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Consider two bodies B_1, B_2 in ordinary space containing the point z and with positive volumes $V(B_i)$. For any unit vector w let $A(B_i, w)$ be the area of the intersection of B_i with the plane normal to w and through z.

Investigations on area lead to the problem, see [2]:

If B_1, B_2 are convex and have z as center, does $A(B_1, w) > A(B_2, w)$ for all w imply $V(B_1) > V(B_2)$?

This very simple sounding problem is still open. One would expect an affirmative answer, even if convexity is replaced by starshapedness with respect to z, or when the condition that z be the center of the B_i is dropped. Here we give examples which show that both assumptions are essential:

(*) *There are bodies of revolution B_1, B_2 starshaped with respect to their common center z with the property*

$$A(B_1, w) > A(B_2, w) \quad \text{for all } w, \quad \text{but } V(B_1) < V(B_2). \tag{1}$$

There are convex bodies of revolution B_i containing z and satisfying (1).

For the construction introduce geographic coordinates u, θ, $-\pi/2 \leqq u \leqq \pi/2$, $0 \leqq \theta < 2\pi$ on the unit sphere S^2. Let $f(u, \theta)$ be a rotation invariant, even, continuous, function on S^2, *i.e.*,

$$f(u, \theta) = f(u), \qquad f(u) = f(-u). \tag{2}$$

Putting $g(s, v) = f[\arcsin(\sin v \cdot \sin s)]$, the value of the integral of f with respect to the arc length over a great circle forming the angle v, $0 \leqq v \leqq \pi$, with the equator $u = 0$ is, because of (2),

$$I_{\pi - v} = I_v = \int_0^{2\pi} g(s, v)ds = 2\int_0^{\pi} g(s, v)ds$$

$$= 2\int_0^{\pi/2} g(s, v)ds + 2\int_{\pi/2}^{\pi} g(\pi - s, v)ds = 4\int_0^{\pi/2} g(s, v)ds, \qquad 0 \leqq v \leqq \pi/2.$$

In this v-interval $\sin v \cdot \sin s$ increases for each s, $0 < s \leqq \pi/2$, hence

$$I_v \geqq I_{\pi/2}, \text{ for } 0 \leqq v \leqq \pi, \text{ if } f(u) \text{ decreases in } [0, \pi/2]. \tag{3}$$

In spherical coordinates r, u, θ with z as the point $r = 0$ we define bodies B_h as bounded by $r = F(u, h) = e^{2|u|} + h(\cos 2u + 2^{-1})e^{-2|u|}$, $|u| \leqq \pi/2$. Then $F(u, h)$ satisfies (2) as function of u, hence B_h is star shaped with respect to z, has z as center and is a surface of revolution.

$$V(B_h) = 3^{-1}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} F^3(u, h)\cos u du = 4\pi 3^{-1}\int_0^{\pi/2} F^3(u, h)\cos u du,$$

$$V'(B_0) = \left.\frac{dV(B_h)}{dh}\right|_{h=0} = 4\pi\int_0^{\pi/2} F^2(u, 0)(\cos 2u + 2^{-1})e^{-2u}du$$

$$= 4\pi\int_0^{\pi/2} e^{2u}(\cos 2u + 2^{-1})du = \pi(-e^{\pi} - 1) + \pi(e^{\pi} - 1) = -2\pi.$$

Thus $V(B_h) < V(B_0)$ for small $h > 0$.

We want to show that $A(B_h, w) > A(B_0, w)$ for small h and all w. Instead of w we may use the previous angle v, because B_h is a body of revolution, so that $A(B_h, v)$ is the area of the intersection of B_h with a plane intersecting the plane $u = 0$ at the angle v. Then as above

$$A(B_h, \pi - v) = A(B_h, v) = 2\int_0^{\pi/2} F^2[\arcsin(\sin v \cdot \sin s), h]ds, \quad 0 \leqq v \leqq \pi/2,$$

$$A'(B_0, v) = \frac{dA(B_h, v)}{dh}\bigg|_{h=0} = 4\int_0^{\pi/2} [\cos 2 \arcsin(\sin v \cdot \sin s) + 2^{-1}]ds.$$

Since $f(u) = \cos 2u + 2^{-1}$ decreases for $0 \leqq u \leqq \pi/2$ we conclude from (3) that $A'(B_0, v) \geqq A'(B_0, \pi/2) = \pi$ for $0 \leqq v \leqq \pi$. It follows that $A(B_h, v) > A(B_0, v)$ for small h and all v, which proves the first part of (*).

The second part is a consequence. For B_h is symmetric with respect to $u = 0$. The body B_h^* bounded by $r = F(u, h)$ for $0 < u \leqq \pi/2$ and the disk $0 \leqq r \leqq F(0, h)$, $u = 0$, is convex for small h. This is obvious for B_0 because $r = F(u, 0) = e^{2u}$, $0 \leqq u \leqq \pi/2$, is part of a spiral, and continuity shows the convexity of B_h for small $h > 0$.

On the other hand

$$v(B_h^*) = V(B_h)/2, \qquad A(B_h^*, v) = A(B_h, v)/2 \text{ for } v \neq 0, \pi,$$

$$A(B_h^*, v) = A(B_h, v) \qquad \text{for } v = 0, \pi.$$

This proves the second part of (*). In spite of the discontinuity of $A(B_h^*, v)$ at $v = 0$ the relations (1) remain valid if z is replaced by a point on the normal to the disk at its center, inside B_h^* and close to that center.

For any two bodies which are star shaped with respect to z and have z as center, $A(B_1, w) = A(B_2, w)$ for all w, implies $B_1 = B_2$; hence all the more $V(B_1) = V(B_2)$. This fact, remarkable in view of our example, follows from a theorem of Funk [3]; see also [2].

If B_1 is an ellipsoid with center z and B_2 is any body containing z, then $A(B_1, w) < A(B_2, w)$ does imply $V(B_1) < V(B_2)$;

See [1]. The discussion there assumes that B_2 is convex, but the result of [1] on which this discussion is based, extends to very general sets B_2.

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ON THE MAXIMUM MODULUS OF POLYNOMIALS*

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1. Introduction. Let $p(z)=a_0+a_1z+\cdots+a_nz^n$ be a polynomial of degree n in the complex variable z. Let $M(p, r)=[\max |p(z)|, |z|=r]$. Let P denote the set of p such that $M(p, 1)=1$. Every p can be normalized by dividing it by $M(p, 1)$ so that $p/M(p, 1)\in P$. Let Q be the set of $p\in P$ such that $p(z)\neq 0$ for $|z|<1$. The following results concerning the size of $M(p, r)$ for $p\in P$ and $p\in Q$ are known.

THEOREM A. (*S. Bernstein, cf.* [1]). *If $p\in P$ then $M(p, R)\leqq R^n$ for $R\geqq 1$, with equality only for $p(z)=\lambda z^n$, $|\lambda|=1$.*

THEOREM B. (*Ankeny and Rivlin* [2]). *If $p\in Q$ then $M(p, R)\leqq (1+R^n)/2$ for $R\geqq 1$ with equality only for $p(z)=(\lambda+\mu z^n)/2$, $|\lambda|=|\mu|=1$.*

THEOREM C. (*Zarantonello-Varga* [3]). *If $p\in P$ then $M(p, r)\geqq r^n$ for $r\leqq 1$ with equality only for $p(z)=\lambda z^n$, $|\lambda|=1$.*

A much simpler proof of Theorem C than is given in [3] is implicit in [1].

It is our purpose here to present a result that stands in relation to Theorem C as Theorem B does in relation to Theorem A. That is, we give an improved bound from below for $M(p, r)$ in the case $p\in Q$. This is done in Section 2 in Theorem 1.

In Section 3 we obtain as a consequence of Theorem 1 a fairly sharp bound (from both above and below) on the maximum modulus of polynomials having all their zeros on the unit circle. Other consequences of Theorem 1 are also discussed.

2. THEOREM 1. *If $p\in Q$ then $M(p, r)\geqq ((1+r)/2)^n$ for $r\leqq 1$, with equality only for $p(z)=((\lambda+\mu z)/2)^n$, $|\lambda|=|\mu|=1$.*

Proof. Suppose $p\in Q$. Let $z_j=R_je^{i\phi_j}$, $j=1, \cdots, n$, be the zeros of p. Since $p\in Q$, $R_j\geqq 1$, $j=1, \cdots, n$. Suppose that $|p(e^{i\alpha})|=M(p, 1)=1$; then for $r<1$ (when $r=1$ there is nothing to prove),

$$M(p, r)=M(p, r)/M(p, 1)\geqq |p(re^{i\alpha})|/|p(e^{i\alpha})|$$

and

$$\frac{|p(re^{i\alpha})|}{|p(e^{i\alpha})|}=\prod_{j=1}^{n}\frac{|re^{i\alpha}-R_je^{i\phi_j}|}{|e^{i\alpha}-R_je^{i\phi_j}|}. \tag{1}$$

We claim now that

$$\left|\frac{re^{i\alpha}-R_je^{i\phi_j}}{e^{i\alpha}-R_je^{i\phi_j}}\right|\geqq\frac{r+R_j}{1+R_j}, \qquad j=1, \cdots, n, \tag{2}$$

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with equality holding if, and only if, $\phi_j=\alpha-\pi$. This may be established by minimizing the function

$$h(\phi) = \frac{|re^{i\alpha} - R_je^{i\phi}|^2}{|e^{i\alpha} - R_je^{i\phi}|^2} - \frac{(r + R_j)^2}{(1 + R_j)^2}$$

for $0\leqq\phi<2\pi$. An easy computation gives

$$h(\phi) = \frac{2R_j(1 - r)(R_j^2 - r)(1 + \cos(\alpha - \phi))}{|e^{i\alpha} - R_je^{i\phi}|^2(1 + R_j)^2},$$

and since $r<1$ and $R_j\geqq 1$ we have $h(\phi)\geqq 0$ and $h(\phi)=0$, if, and only if, $\cos(\alpha-\phi)=-1$, or $\alpha-\phi=\pi$, or $\phi=\alpha-\pi$. Thus (2) is established.

Furthermore, since $R_j\geqq 1$ we have

$$(r + R_j)/(1 + R_j) \geqq \tfrac{1}{2}(1 + r), \qquad j = 1, \cdots, n \tag{3}$$

with equality if, and only if, $R_j=1$. To establish (3) we need only note that since $r<1$ the function $(r+x)/(1+x)$ is strictly monotone increasing for $x>0$.

Now, using inequalities (2) and (3) in (1) we obtain

$$M(p, r) \geqq ((1 + r)/2)^n \tag{4}$$

and equality holds in (4) if, and only if, $R_j=1, \phi_j=\alpha-\pi, j=1, \cdots, n$. Thus we must have $p(z)=\gamma(z+e^{i\alpha})^n$. But $M(p, 1)=1$; hence $|\gamma(2e^{i\alpha})|^n=1$ and $\gamma=\beta 2^{-n}$ where $|\beta|=1$. The proof of Theorem 1 is complete.

3. By combining Theorem 1 with Theorem B we can obtain fairly sharp bounds from above and below for a polynomial all of whose zeros have absolute value one. More precisely, we have

THEOREM 2. *If $p\in P$, and $p(z)=0$ implies $|z|=1$, then*

$$\{\tfrac{1}{2}(1 + \rho)\}^n \leqq M(p, \rho) \leqq \tfrac{1}{2}(1 + \rho^n), \qquad \rho \geqq 0.$$

Proof. In view of Theorems 1 and B we need only to show (i) $M(p, \rho)\leqq(1+\rho^n)/2$, $\rho<1$ and (ii) $M(p, \rho)\geqq((1+\rho)/2)^n$, $\rho>1$.

To this end put $q(z)=z^np(1/z)$. Then $q\in Q$ and so:

(i) by Theorem B, if $R=1/\rho>1$

$$\tfrac{1}{2}(1 + R^n) \geqq M(q, R) = R^nM(p, \rho);$$

hence $M(p, \rho)\leqq\frac{1}{2}(1+\rho^n)$.

(ii) by Theorem 1, if $r=1/\rho<1$, $((1+r)/2)^n\leqq M(q, r)=r^nM(p, \rho)$; hence

$$M(p, \rho) \geqq \{\tfrac{1}{2}(1 + \rho)\}^n.$$

Another consequence of Theorem 1 is that for $p\in Q$ we have, taking $r=0$,

$$|p(0)| = |a_0| \geqq 2^{-n}.$$

Indeed, suppose $p(0)\neq 0$, and let a be the absolute value of a zero of smallest

absolute value of p (we are not assuming $p \in Q$ here). Then putting $q(z) = p(az)$ we see that $q/M(q, 1) \in Q$ and so $|q(0)| \geqq M(q, 1)2^{-n}$ or

$$|p(0)| = |a_0| \geqq 2^{-n} M(p, a). \tag{5}$$

Equation (5) may be expressed as

THEOREM 3. *If $M(p, \rho) > |a_0| 2^n$ for some $\rho > 0$ then the polynomial $p(z)$ has at least one zero in $|z| < \rho$.*

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A NOTE ON WALSH-FOURIER SERIES

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Let $F(t)$ be a function such that $|e^{-st}F(t)| < Ae^{-(a-\sigma)t}$ where a and A are constants. Then $\int_0^\infty e^{-st}F(t)dt$ exists for $R(s) > a$ and defines a new function $f(s)$ called the Laplace transform of the determining function $F(t)$. If one uses the Walsh function as the object function the following infinite product expansion may be proved.

$$\prod_{k=0}^{\infty} \left\{ \frac{2}{1 + x^{1/2^k}} \right\} = \frac{2 \log x}{1 - x^2}, \qquad 0 < x < 1. \tag{1}$$

Although this formula may be proved directly, I think the following derivation is enlightening.

Since the Walsh functions may not be familiar to many, I shall define them by using the Rademacher functions:

$$\phi_0(x) = 1 \quad (0 \leqq x < 1/2), \quad \phi_0(x + 1) = \phi_0(x),$$
$$\phi_0(x) = -1 \quad (1/2 \leqq x < 1); \qquad \phi_n(x) = \phi_0(2^n x), \quad n = 1, 2, \cdots.$$

These Rademacher functions may now be used to define the Walsh functions as follows:

$$\psi_0(x) = 1; \qquad \psi_n(x) = \phi_{n_1}(x) \cdots \phi_{n_r}(x)$$

for $n = 2^{n_1} + \cdots + 2^{n_r}$, where the n_i are uniquely determined by $n_{i+1} < n_i$. Using the above definition of the Walsh function, one may prove many interesting properties such as

$$\psi_{2n+1}(x) = \psi_{2n}(x)\psi_1(x), \tag{A}$$

$$\psi_{2n}(x) = \psi_n(2x), \tag{B}$$

(C) $$\psi_{2n}(x + 1/2) = \psi_{2n}(x).$$

Walsh has proved that the sequence $\{\psi_n(x)\}$ forms a complete orthonormal set, and every periodic function $g(x) \in L(0, 1)$ has a Walsh-Fourier series

$$g(x) \sim C_0 + C_1\psi_1(x) + \cdots + C_n\psi_n(x) + \cdots,$$

where

$$C_n = \int_0^1 \psi_n(x)g(x)dx; \qquad n = 0, 1, \cdots.$$

Consider the Laplace transform of $\psi_n(t)$ which certainly has period 1; then for $R(s) > 0$

$$f_n(s) = \int_0^\infty e^{-st}\psi_n(t)dt = \sum_{k=0}^\infty \int_k^{k+1} e^{-st}\psi_n(t)dt = C_n(s) \sum_{k=0}^\infty e^{-ks} = C_n(s)\{1/(1 - e^{-s})\}$$

where $C_n(s)$ is the nth Walsh-Fourier coefficient of the function which is equal to e^{-st} for $0 \leqq t < 1$ and which has period 1. Since the function is of bounded variation and continuous on the right one may write $e^{-st} = \sum_{n=0}^\infty C_n(s)\psi_n(t)$ by a theorem due to J. L. Walsh stated in [1]. The Walsh-Fourier coefficients for e^{-st} give rise to the following interesting recurrence formulas:

(a) $$C_{2n}(s) = 1/2(1 + e^{-s/2})C_n(s/2),$$

(b) $$C_{2n+1}(s) = 1/2(1 - e^{-s/2})C_n(s/2),$$

(c) $$C_{2n+1}(s) = \{(1 - e^{-s/2})/(1 + e^{-s/2})\}C_{2n}(s),$$

(d) $$C_n(s) = (1/s)(1 - e^{-s})P_n(s),$$

where

$$P_n(s) = \prod_{j=1}^r \frac{1 - \exp(-s/2^{n_j+1})}{1 + \exp(-s/2^{n_j+1})}; \quad P_0(s) \equiv 1,$$

$$n = 2^{n_1} + \cdots + 2^{n_r}; \qquad n_1 > \cdots > n_r \geqq 0.$$

Recurrence formulas (a) and (b) are worthy of proof, because they make use of the properties of the Walsh function. Applying property (B); one may derive equation (a) as follows:

$$C_{2n}(s) = \int_0^1 e^{-st}\psi_{2n}(t)dt = \int_0^1 e^{-st}\psi_n(2t)dt = 1/2 \int_0^2 e^{-(s/2)t}\psi_n(t)dt$$

$$= 1/2 \int_0^1 e^{-(s/2)t}\psi_n(t)dt + 1/2 \int_1^2 e^{-(s/2)t}\psi_n(t)dt.$$

Substituting $t = u + 1$ into the second integral and remembering that $\psi_n(t)$ has period one, we obtain

$$C_{2n}(s) = 1/2C_n(s/2) + 1/2e^{-(s/2)}\int_0^1 e^{-(s/2)u}\psi_n(u)du = 1/2(1 + e^{-(s/2)})C_n(s/2).$$

Applying properties (A), (B) and $\psi_1(t) = \phi_0(t)$; equation (b) may be derived as follows:

$$C_{2n+1}(s) = \int_0^1 e^{-st}\psi_{2n+1}(t)dt = \int_0^1 e^{-st}\psi_{2n}(t)\psi_1(t)dt$$

$$= \int_0^{1/2} e^{-st}\psi_n(2t)dt - \int_{1/2}^1 e^{-st}\psi_{2n}(t)dt.$$

substituting $t = (1/2)u$ into the first integral and $t = u + 1/2$ into the second integral and applying property (C), one obtains

$$C_{2n+1}(s) = 1/2\int_0^1 e^{-(s/2)u}\psi_n(u)du - e^{-(s/2)}\int_0^{1/2} e^{-su}\psi_{2n}(u)du$$

$$= 1/2C_n(s/2) - 1/2e^{-(s/2)}\int_0^1 e^{-(s/2)u}\psi_n(u)du = 1/2(1 - e^{-(s/2)})C_n(s/2).$$

These recurrence formulas may now be used to derive* a general product expansion which reduces to equation (1). Let $f(s, t) = e^{-st}$ $(0 \leqq t < 1)$, $f(s, t+1) = f(s, t)$. Then for all t,

$$f(s, t) = \sum_{n=0}^{\infty} C_n(s)\psi_n(t) = \sum_{n=0}^{\infty} C_{2n}(s)\psi_{2n}(t) + \sum_{n=0}^{\infty} C_{2n+1}(s)\psi_{2n+1}(t)$$

$$= 1/2(1 + e^{-s/2})\sum_{n=0}^{\infty} C_n(s/2)\psi_n(2t) + \tfrac{1}{2}(1 - e^{-s/2})\psi_1(t)\sum_{n=0}^{\infty} C_n(s/2)\psi_n(2t)$$

$$= 1/2(1 + e^{-s/2})\{1 + \lambda(s)\psi_1(t)\}f(s/2, 2t).$$

If we define $g(s, t) = \{s/(1 - e^{-s})\}f(s, t)$, this functional equation reduces to

$$g(s, t) = [1 + \lambda(s)\psi_1(t)]g(s/2, 2t), \tag{2}$$

where $\lambda(s) = \tanh(s/4)$. By iterating (2), we get

$$g(s, t) = \prod_{k=0}^{n-1}[1 + \lambda(s/2^k)\psi_{2^k}(t)]\cdot g(s/2^n, 2^n t).$$

It is easy to see that $g(s/2^n, 2^n t) \to 1$ uniformly in t, so

$$g(s, t) = \prod_{k=0}^{\infty}(1 + \lambda(s/2^k)\psi_{2^k}(t)).$$

Putting $s = -2\log x$, we get

* The referee suggested this proof for (1).

$$\frac{-2x^{2t}\log x}{1-x^2} = \prod_{k=0}^{\infty}\left\{\frac{2}{1+x^{b_k(t)/2^k}}\right\} \qquad (0 \leqq t < 1).$$

Substituting $t=0$, we obtain equation (1).

An alternative proof of the product expansion (1) may also be obtained as follows:

$$e^{-st} = \sum_{n=0}^{\infty} C_n(s)\psi_n(t) \qquad (0 \leqq t < 1, s \neq 0)$$

$$= \sum_{n=0}^{\infty} \left\{(1 - e^{-s})/s\right\} P_n(s)\psi_n(t)$$

$$= (1/s)(1-e^{-s}) \sum_{n=0}^{\infty} \left\{\tanh(s/2^{n_1+2}) \cdots \tanh(s/2^{n_r+2})\right\}\psi_n(t).$$

Putting $t=0$, one obtains

$$1 = (1/s)(1-e^{-s}) \sum_{n=0}^{\infty} P_n(s)$$

which reduces to

$$s/(1-e^{-s}) = 1 + \tanh(s/2^2) + \tanh(s/2^3) + \tanh(s/2^3)\cdot\tanh(s/2^2)$$
$$+ \tanh(s/2^4) + \cdots.$$

Hence

$$s/(1-e^{-s}) = \prod_{k=0}^{\infty}(1+\tanh(s/2^{k+2})), \qquad R(s) > 0. \tag{3}$$

Using the transformation $s=-2\log x$ for $0<x<1$, one may easily show that

$$s/(1-e^{-s}) = -2\log[x/(1-x^2)] \quad \text{and} \quad 1+\tanh(s/2^{k+2}) = 2/(1+x^{1/2^k}).$$

Substituting into (3), we obtain

$$-2\log[x/(1-x^2)] = \prod_{k=0}^{\infty}\left\{2/(1+x^{1/2^k})\right\} \qquad 0 < x < 1.$$

Note that

$$\lim_{x\to 1-}\left\{-2\log[x/(1-x^2)]\right\} = 1.$$

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PARALLELOGRAMS INSCRIBED IN CONVEX CURVES

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There is a wide class of theorems concerning convex curves of the form: Within any convex curve of a certain type can be found a convex curve of another type.

For example each convex curve F, surrounding a closed set K of area A and minimum width d, contains a triangle of area $2\sqrt{3}A/4\pi$, a radially symmetric convex set of area $2A/3$ ([1] p. 116), a circle of radius $d/3$ ([1] p. 112), a curve of constant width $d/(3-\sqrt{3})$ ([2] p. 141). Each convex curve F of constant width contains a radially symmetric convex curve of area $(.84)A$ ([2] p. 131). Some of these "best possible" results have been extended to higher dimensions, some have not. In the latter class belongs*

THEOREM 1. *If F is a convex curve surrounding a closed set K of area A then there is a parallelogram inscribed† in F of area $A/2$. Moreover there is no parallelogram of area strictly larger than $A/2$ inscribed in F if and only if F is a triangle.*

Proof. Assume first that F is neither a triangle nor a quadrilateral. Choose four support lines of F enclosing a polygon of the smallest possible area. Since F is not a triangle it is easily seen that these four lines are distinct. In addition the midpoint of each of the four sides of this quadrilateral of minimal area must lie in F. Were this not the case, a simple continuity argument could be used to construct a quadrilateral of smaller area containing K. Thus the parallelogram with the four midpoints of a minimal quadrilateral as vertices lies in K. Thus, since K is not a quadrilateral, K contains a parallelogram with area strictly larger than $A/2$.

Next assume that F is a triangle. Since a parallelogram has four vertices two lie on one side of F. A straight forward examination of such parallelograms shows that the largest has area $A/2$.

Finally let $ACBD$ be a convex quadrangle other than a trapezoid such that $AD, BC=P$ and $AC, BD=Q$ are the exterior diagonal points and C is inside the diagonal triangle. The vertex C is then between A and Q and also between B and P. We now make D the origin of a system of oblique Cartesian coordinates and assign coordinates as follows: $A(a, 0)$, $B(0, b)$, $C(x_0, y_0)$, where all nonzero values are assumed positive. The points P and Q are also located on the positive semiaxes. The condition of convexity is expressed as

$$x_0/a + y_0/b > 1. \tag{1}$$

For the proof it seems necessary to distinguish between two cases.

Case 1. We assume that $2x_0>a$, $2y_0>b$. The points $(2x_0, 0)$, $(0, 2y_0)$, and D are the vertices of a triangle which contains the given quadrangle. Now the

* This problem arose in conversation with Donald C. Benson.

† The theorem remains valid if "inscribed in F" is replaced by "contained in K."

parallelogram with sides $x=0$, $x=x_0$, $y=0$, $y=y_0$ has area equal to one half that of the triangle and hence more than half that of the quadrangle.

Case 2. Assuming that $2x_0 \leqq a$, we find from (1) that $2y_0 > b$. We draw the parallels to the axes through the midpoint of AQ. These together with the axes determine a parallelogram whose area is half the area of triangle DAQ. But the original quadrangle is inscribed in the latter. Again the desired conclusion follows.

The trapezoid can be dealt with by making P an ideal point. Otherwise the proofs remain unchanged. This concludes the proof.

Macbeath has shown ([3] pp. 59–60) that if K is a convex set of volume V in n-dimensional space then K contains a parallelotope of volume V/n^n and is contained in one of volume $n!\,V$. From his argument of ([3] p. 58) and Theorem 1 of this paper one deduces

THEOREM 2. *If K and K' are convex sets in the plane with nonzero areas A and A' respectively then there is an affine image of K', $\sigma K'$, contained in K, such that area $\sigma K'$ is greater than $A/4$, and also an affine image $\tau K'$, containing K such that area $\tau K'$ is less than $4A$.*

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CIRCULANTS AND THEIR GROUPS*

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Let the elements in the first row of the symbol for a determinant of order n be $x_0, x_1, \cdots, x_{n-1}$. If we permute the elements cyclically, beginning with x_{n-1} in the second row, x_{n-2} in the third row etc., the result is a circulant [1], C_n, of order n. That is, $C_n = |x_{r-c}|$ $r, c = 0, 1, \cdots, n-1$. It is known [2] that every rational function belongs to some group. The purpose of this note is to derive the group to which C_n belongs.

Let the permutation $P = (x_0 x_1 \cdots x_{n-1})$ and apply P^α to the factored form of

$$C_n = \prod_{j=0}^{n-1} \sum_{k=0}^{n-1} \epsilon^{jk} x_k, \qquad \epsilon = \exp(2\pi i/n).$$

The factor $r_j = \sum_{k=0}^{n-1} \epsilon^{jk} x_k$ becomes

$$\sum_{k=0}^{n-1} \epsilon^{jk} x_{k+\alpha} = \epsilon^{-j\alpha} \sum_{k=0}^{n-1} \epsilon^{j(k+\alpha)} x_{k+\alpha} = \epsilon^{-j\alpha} r_j.$$

* Presented to the Southeastern Section of the Association, March 15, 1958.

Now $\prod_{j=0}^{n-1} \epsilon^{-j\alpha} = \epsilon^{-n\alpha(n-1)/2}$ so that P^α leaves C_n unaltered for every α if and only if n is odd. If n is even, P^α leaves C_n unaltered if and only if α is even. Hence, if n is odd, there are at least n permutations which leave C_n unaltered by changing each r_j into a constant multiple of itself and at least $n/2$ such permutations if n is even.

Next let

$$Q = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{i_0} & x_{i_1} & \cdots & x_{i_{n-1}} \end{pmatrix} = \begin{pmatrix} x_k \\ x_{i_k} \end{pmatrix}$$

be a permutation that leaves C_n unaltered by changing each r_j into a constant multiple of itself. Since the constants are powers of ϵ, we have

$$\sum_{k=0}^{n-1} \epsilon^{jk} x_{i_k} = \epsilon^{tj} r_j.$$

From this identity we obtain the congruences

$$\alpha j - t_j \equiv i_\alpha j \pmod{n}, \qquad \alpha = 0, 1, \cdots, n-1.$$

By successive elimination we find that $(i_{\alpha+1} - i_\alpha)j \equiv j \pmod{n}$. If $(j, n) = 1$, we obtain $i_\alpha = i_{0+\alpha}$. Also

$$\prod_{j=0}^{n-1} \epsilon^{tj} = \epsilon^{-ni_0(n-1)/2} = 1$$

if n is odd or both n and i_0 are even.

LEMMA. *The largest permutation group leaving C_n unaltered by changing each r_j into a constant multiple of itself is $\{P\}$ if n is odd and $\{P^2\}$ if n is even.*

Now let G be the largest group on $x_0, x_1, \cdots, x_{n-1}$ which leaves C_n unaltered and let K be the group of permutations induced on $r_0, r_1, \cdots, r_{n-1}$. Then the subgroup $H = \{P\}$, n odd, and $\{P^2\}$, n even, corresponds to the identity in K and is invariant in G. Thus, if U is any permutation of G, then $U^{-1}PU = P^\alpha$, where $(\alpha, n) = 1$. We may represent U in the form

$$\begin{pmatrix} x_j \\ x_{\alpha j} \end{pmatrix}$$

and easily show that the permutation induced on the r's may be represented in the form

$$\begin{pmatrix} r_j \\ r_{j/\alpha} \end{pmatrix}.$$

Hence, if n is odd, there are at least n permutations, $\{P\}U$, which leave C_n unaltered by changing each r_j into a multiple of $r_{j/\alpha}$. If there were more than these n, we could, by multiplying each of them on the right by U^{-1}, obtain more

than n permutations which leave C_n unaltered by changing each r_j into a constant multiple of itself. This contradicts the lemma.

THEOREM. *The order of G is $n\phi(n)$ if n is odd and $(n/2)\phi(n)$ if n is even.*

COROLLARY 1. *A circulant of prime order p belongs to the metacyclic* [3] *group of degree p.*

COROLLARY 2. *An isomorphism is established by the correspondence $\{H\}U$ and V, where U is a permutation on the x's and V permutes the r's in the corresponding inverse order.*

COROLLARY 3. *A circulant of order n is a symmetric function if and only if $n=3$.*

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SOME INFINITE SERIES FOR $\zeta(n+1)$

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Briggs, Chowla, Kempner, and Mientka [1] have shown that

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\right\}, \tag{1}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n}\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right\}. \tag{2}$$

By the same methods it can be shown that

$$\zeta(n+1) = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^2}, \qquad \text{for } n \geqq 1, \tag{3}$$

where

$$s_{n-1}(k_n) = \sum_{k_{n-1}=n-1}^{k_n-1} \frac{s_{n-2}(k_{n-1})}{k_n}, \qquad s_0(k_1) = 1. \tag{4}$$

If $n=2$, (3) reduces to (1). Also, using the same notations,

$$\sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^3} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}}, \qquad \text{for } n \geqq 1, \tag{5}$$

which reduces to (2) for $n=1$. Finally it can be shown that

$$\zeta(n+1) = \sum_{m=1}^{\infty} \frac{1}{m^n} \sum_{k=1}^{m} \frac{1}{k} - \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^n}, \qquad \text{for } n \geqq 2. \tag{6}$$

Since, for $|x|<1$,

$$\log\left(\frac{1}{1-x}\right)=\sum_{k=1}^{\infty}\frac{x^k}{k},$$

it can be shown by induction that

$$\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n}x^{k_n}. \tag{7}$$

To do this consider

$$\left(\frac{1}{1-x}\right)\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n=\left\{\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)x^{k_n}}{k_n}\right\}\left\{\sum_{k=0}^{\infty}x^k\right\}$$

$$=\sum_{m=n}^{\infty}\left(\sum_{k_n=n}^{m}\frac{s_{n-1}(k_n)}{k_n}\right)x^{m+1},$$

so that

$$\frac{1}{(n+1)!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^{n+1}=\sum_{m=n}^{\infty}\frac{1}{m+1}\left(\sum_{k_n=n}^{m}\frac{s_{n-1}(k_n)}{k_n}\right)x^{m+1}$$

$$=\sum_{k_{n+1}=n+1}^{\infty}\frac{s_n(k_{n+1})}{k_{n+1}}x^{k_{n+1}}$$

Consider [3]

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{e^t-1}\,dt,$$

and let $e^{-t}=1-x$, $s=n+1$. Thus

$$\zeta(n+1)=\int_0^1\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n\frac{dx}{x}$$

$$=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n}\int_0^1 x^{k_n-1}dx=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n^2},$$

which is (3). Since

$$\frac{n!}{k^{n+1}}=\int_0^{\infty}t^ne^{-kt}dt,$$

we have by summing on k, setting $e^{-t}=1-x$, and using (7) and (3)

$$\sum_{k=1}^{m}\frac{1}{k^{n+1}}=\int_0^1\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n[1-(1-x)^m]\frac{dx}{x}$$

$$= \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n} \left\{ \int_0^1 x^{k_n-1}dx - \int_0^1 x^{k_n-1}(1-x)^m dx \right\}$$

$$= \zeta(n+1) - \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n} \cdot \frac{\Gamma(m+1)\Gamma(k_n)}{\Gamma(m+1+k_n)} .$$

Hence

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)(k_n-1)!}{k_n} \sum_{m=1}^{\infty} \frac{(m-1)!}{(k_n+m)!} .$$

Let $f(m)=(m-1)!/k_n(m+k_n-1)!$; so that $f(1)=1/(k_n)!k_n$, $f(m)-f(m+1)=(m-1)!/(m+k)!$ and $\sum_{m=1}^{\infty}(m-1)!/(k_n+m)!=f(1)$; and thus (5) is obtained:

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^3} .$$

Finally, let $R_{n+1}(m+1)=\sum_{k=m+1}^{\infty} 1/k^{n+1}$, $s_1(1)=0$ and consider

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \sum_{j=1}^{m-1} \frac{1}{j} \right) R_{n+1}(m+1)$$

$$= \sum_{m=1}^{\infty} [s_1(m+1) - s_1(m)] R_{n+1}(m+1)$$

$$= \sum_{m=1}^{\infty} s_1(m)[R_{n+1}(m) - R_{n+1}(m+1)]$$

$$= \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \frac{1}{m} \right) \frac{1}{m^{n+1}}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^{n+1}} \sum_{k=1}^{m} \frac{1}{k} - \zeta(n+2)$$

which is equivalent to (6).

Formula (6) can also be written in the form

$$\zeta(n+1) = \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{k^{n-1}+m^{n-1}}{(mk)^n} - \zeta(n) \right), \qquad \text{for } n \geqq 2.$$

It does not appear obvious that there is a direct connection between these formulas and those of Brun [2] (setting $s=n+1$),

$$\zeta(n+1) = \frac{n+1}{n} + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2^{rn}}{(2^r+m)^{n+1}},$$

$$\zeta(n+1) = \frac{n+1}{n} + \sum_{m=3}^{\infty} \frac{(-1)^{m-1}}{m^{n+1}} \sum_{k=0}^{[\sqrt{(m-1)}]} 2^{kn}.$$

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A VECTOR SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

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In the following note we adopt the summation convention for repeated indices; the indices r and j will always have the range $1, \cdots, n$. Let $e_1, \cdots, e_n$ be an orthonormal base system of an n-dimensional vector space.* We define the vector product of the $n-1$ vectors $V_i = v_i^r e_r$, $i = 1, \cdots, n-1$, to be the vector obtained by expanding the determinant,

$$V = \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ v_1^1 & v_1^2 & \cdots & v_1^n \\ \cdot & \cdot & \cdots & \cdot \\ v_{n-1}^1 & v_{n-1}^2 & \cdots & v_{n-1}^n \end{vmatrix} \tag{1}$$

in terms of elements in the first row. For any vector U, $U \cdot V = 0$ if and only if the vectors $U, V_1, \cdots, V_{n-1}$ are linearly dependent. Now consider the n points $(b_j^1, \cdots, b_j^n)$ which determine a hyperplane

$$A_r x^r = M, \qquad M \neq 0 \text{ if } \det b_j^r \neq 0. \tag{2}$$

If P and Q are any two points whose coordinates, c^r and d^r, satisfy (2), we say that P and Q lie in the plane, and define the vector $[P, Q]$ by $[P, Q] = (d^r - c^r)e_r$. We also define the coefficient vector of (2) by $C = A_r e^r$ ($e^r = e_r$ since the system is orthonormal). The vector $[P, Q]$ is orthogonal to the coefficient vector, C, for $[P, Q] \cdot C = M - M = 0$.

THEOREM 1. *The points* $(b_j^1, \cdots, b_j^n)$ *whose* $\det b_j^r \neq 0$ *determine at least one set of* $n-1$ *linearly independent vectors in the hyperplane determined by the points.*

Proof. (constructive)

$$0 \neq \det b_j^r = \begin{vmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 - b_1^1 & b_2^2 - b_1^2 & \cdots & b_2^n - b_1^n \\ \cdot & \cdot & \cdots & \cdot \\ b_n^1 - b_1^1 & b_n^2 - b_1^2 & \cdots & b_n^n - b_1^n \end{vmatrix}. \tag{3}$$

* See, for example, Louis Brand, Vector Analysis, New York, 1957, Ch. 9.

Therefore the vectors $(b_s^r - b_1^r)e_r$, $s = 2, \cdots, n$, which lie in the plane, are linearly independent.

THEOREM 2. *The vector product, V, of $n-1$ vectors, V_i, in the hyperplane $A_r x^r = M$, is orthogonal to any vector, U, in the plane. If the $n-1$ vectors, V_i, are linearly independent, then U is a linear combination of these vectors.*

Proof. Let $V_i = v_i^r e_r$, $i = 1, \cdots, n-1$, and $U = u^r e_r$. Since $A_r v_i^r = 0$, $A_r u^r = 0$, and not every $A_r = 0$,

$$U \cdot V = \begin{vmatrix} u^1 & u^2 & \cdots & u^n \\ \overset{1}{v_1} & \overset{2}{v_1} & \cdots & \overset{n}{v_1} \\ \cdot & \cdot & \cdots & \cdot \\ \overset{1}{v_{n-1}} & \overset{2}{v_{n-1}} & \cdots & \overset{n}{v_{n-1}} \end{vmatrix} = 0.$$

Thus U, $V_1, \cdots, V_{n-1}$ are linearly dependent. Hence if the V_i are linearly independent, then U is a linear combination of the V_i.

THEOREM 3. *The coefficient vector, C, of the hyperplane, $A_r x^r = M$, determined by n points whose* det $b_j^r \neq 0$ *is a scalar product of the vector product of $n-1$ linearly independent vectors in the plane.*

Proof. Denote the vectors $b_j^r e_r$ by b_j. Then we have for $s = 2, \cdots, n$, $C \cdot (b_s - b_1) = 0$, and $V \cdot (b_s - b_1) = 0$ by Theorem 2. Further $C \cdot b_1 = M \neq 0$, and let $V \cdot b_1 = K$. Since the vectors, V_i, which define V are linearly independent by Theorem 2, the second to n-th rows of the nonzero det (3) are linear combinations of the second to n-th rows of det (1); hence $K \neq 0$. Let $b_1 = X_1$ and $b_s - b_1 = X_s$ for $s = 2, \cdots, n$. Then combining the above results we have $[(1/K)V - (1/M)C] \cdot X_i = 0$, $i = 1, \cdots, n$, where the X_i are n linearly independent vectors by the proof of Theorem 1. Hence $(1/K)V - (1/M)C = 0$, or $C = (M/K)V$.

Now consider the n simultaneous linear equations in n variables, x_r in which a_j^r are constants,

$$a_j^r x_r = M, \qquad M \neq 0, \qquad \det a_j^r \neq 0. \tag{4}$$

In (4) we may interpret the $(a_j^1, \cdots, a_j^n)$ as n points and the x_r as the components of the coefficient vector of the hyperplane determined by the points. By the method of Theorem 1 we construct $n-1$ linearly independent vectors whose vector product is a scalar product of the coefficient vector. Thus if the vector product is $V = v^r e_r$, we have $N v^r = x_r$. To find N we substitute in one of the equations (4). If det $a_j^r = 0$, then the vectors $(a_s^r - a_1^r)e_r$, $s = 2, \cdots, n$, are linearly dependent, and Theorem 3 does not apply; the constructed vector product $V = 0$.

Example. Solve the system:

$$\begin{aligned} 2x+\ y+\ z-\ w&=3 \\ x-\ y-\ z+2w&=2 \\ x+2y+\ z-3w&=1 \\ -x+3y-2z+\ w&=6 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4x+\ 2y+2z-\ 2w&=6 \\ 3x-\ 3y-3z+\ 6w&=6 \\ 6x+12y+6z-18w&=6 \\ -x+\ 3y-2z+\ w&=6 \end{aligned}$$

We must solve the right hand set as the method requires all constants on the right to be identical. Hence if one of the equations has a 0 on the right that equation must be combined with another prior to multiplication to obtain an equivalent set as was done above. From the points $A(4, 2, 2, -2)$, $B(3, -3, -3, 6)$, $C(6, 12, 6, -18)$, $D(-1, 3, -2, 1)$ we form the vectors $[B, A]=e_1+5e_2+5e_3-8e_4$, $[C, A]=-2e_1-10e_2-4e_3+16e_4$, $[D, A]=5e_1-e_2+4e_3-3e_4$ or just as well replace $[C, A]$ by $-\frac{1}{2}[C, A]=e_1+5e_2+2e_3-8e_4$. Thus we have

$$V=\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 5 & 5 & -8 \\ 1 & 5 & 2 & -8 \\ 5 & -1 & 4 & -3 \end{vmatrix} = 69e_1+111e_2+78e_4.$$

Hence $x=69N$, $y=111N$, $z=0$, $w=78N$. Substituting in the first equation we find $N=1/57$; hence $x=23/19$, $y=37/19$, $z=0$, $w=26/19$.

The author wishes to express his appreciation for the useful suggestions made by the referee.

ON THE PRODUCT OF TWO UNIFORMLY CONTINUOUS FUNCTIONS ON THE LINE

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1. Many properties of continuous functions defined on a closed interval fail to hold for continuous functions defined on the line. For example, continuity may or may not involve uniform continuity, as the functions x and x^2 illustrate. Even when two functions are uniformly continuous on the line, their product need not be uniformly continuous, as for example, x and $\sin x$. It might be of some interest to investigate the question of the product of two uniformly continuous real functions on the line. In doing so we shall restrict our attention to non-negative functions defined on $[1, \infty)$.

2. Let K denote the class of non-negative uniformly continuous functions defined on $[1, \infty)$. The following lemmas will be useful in the sequel.

LEMMA 2.1. *If* $f(x)\in K$, *then* $\limsup_{x\to\infty} f(x)/x<\infty$.

Proof. Suppose on the contrary that $\limsup f(x)/x=\infty$. Then there exists a sequence of reals $\{x_k\}$ such that $x_k\to\infty$, $x_{k+1}-x_k>1$, and if $c_k=f(x_k)/x_k$ then $c_{k+1}\geqq c_k$ and $c_k\to\infty$. Now for each k, $|f(x_{k+1})-f(x_k)|=c_{k+1}x_{k+1}-c_kx_k \geqq c_k(x_{k+1}-x_k)$. Let $\epsilon=1$, and suppose that δ is arbitrary except that $0<\delta<1$.

There exists an index k_0 such that $c_k > 2/\delta$ when $k > k_0$. Let $k > k_0$, and choose $t_1, t_2, \cdots, t_m$ such that $x_k = t_1 < t_2 < \cdots < t_m = x_{k+1}$, where $\delta/2 < t_{i+1} - t_i < \delta$ for $i = 0, 1, \cdots, m-1$. For some index j it must follow that $|f(t_{j+1}) - f(t_j)| \geqq c_k(t_{j+1} - t_j)$; otherwise

$$|f(x_{k+1}) - f(x_k)| \leqq \sum_{i=0}^{m-1} |f(t_{i+1}) - f(t_i)| < c_k(x_{k+1} - x_k),$$

contrary to hypothesis. Hence $|f(t_{j+1}) - f(t_j)| \geqq c_k(t_{j+1} - t_j) > 2/\delta \cdot \delta/2 = 1$, and $f(x)$ is not uniformly continuous.

LEMMA 2.2. *If $xf(x) \in K$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $x|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$.*

Proof. Let $\epsilon > 0$. Since $xf(x) \in K$, there is by Lemma 2.1 a constant $M > 0$ such that $f(x) < M$. Moreover, there exists a $\delta > 0$ such that $\delta < \epsilon/(2M)$ and $|xf(x) - yf(y)| < \epsilon/2$ when $|x - y| < \delta$. Hence $x|f(x) - f(y)| \leqq |xf(x) - yf(y)| + f(y)|x - y| < \epsilon/2 + M \cdot \epsilon/(2M) = \epsilon$, when $|x - y| < \delta$.

LEMMA 2.3. *If $x^2 f(x) \in K$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $x^2|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$.*

Proof. Let $\epsilon > 0$. Since $x^2 f(x) \in K$, there exists a constant $M > 0$ such that $xf(x) < M$. Moreover there exists a $\delta > 0$ such that $\delta < \text{Min}\,(1, \epsilon/(6M))$ and $|x^2 f(x) - y^2 f(y)| < \epsilon/2$ when $|x - y| < \delta$. Hence

$$\begin{aligned} x^2|f(x) - f(y)| &\leqq |x^2 f(x) - y^2 f(y)| + f(y)|x^2 - y^2| \\ &< \epsilon/2 + yf(y)(2 + 1/y)|x - y| < \epsilon/2 + M \cdot (3\epsilon)/(6M). \end{aligned}$$

3. In approaching the question of the product of two functions in K, consider the class of functions K^* defined as follows: K^* is the class of all functions $f(x)$ such that $f(x) \in K$ and $f(x)g(x) \in K$ whenever $g(x)$ is in K. The following theorem will serve to characterize the class K^*.

THEOREM 3.1. *$f(x) \in K^*$ if and only if $xf(x) \in K$.*

Proof. Since $x \in K$ the necessity is trivial. Conversely, suppose $xf(x) \in K$. Let $\epsilon > 0$ and let $g(x) \in K$. By Lemma 2.1 there exist constants M_1 and M_2 such that $f(x) < M_1$ and $g(x)/x < M_2$. By hypothesis on $g(x)$ and by Lemma 2.2 it is possible to choose $\delta > 0$ such that $|g(x) - g(y)| < \epsilon/(2M)_1$ and $y|f(x) - f(y)| < \epsilon/(2M_2)$ when $|x - y| < \delta$. Hence

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leqq f(x)|g(x) - g(y)| + (g(y)/y)y|f(x) - f(y)| \\ &< M_1 \cdot \epsilon/(2M_1) + M_2 \cdot \epsilon/(2M_2) = \epsilon \end{aligned}$$

when $|x - y| < \delta$. It follows that $f(x) \in K^*$.

Pursuing this line, let K^{**} denote the class of functions $f(x)$ such that $f(x) \in K^*$ and $f(x)g(x) \in K^*$ whenever $g(x) \in K$. K^{**} is a proper subset of K^*

since the function $f(x)\equiv 1$ is in K^* but not in K^{**} because $x\in K$ but $x\cdot 1\notin K^*$. The following theorem, analogous to Theorem 3.1, characterizes the class K^{**}.

THEOREM 3.2. *$f(x)\in K^{**}$ if and only if $x^2f(x)\in K$.*

Proof. By definition of K^{**} and K^*, if $f(x)\in K^{**}$, then $xf(x)\in K^*$; hence $x^2f(x)\in K$. Conversely, suppose $x^2f(x)\in K$. Clearly, if $f(x)g(x)h(x)\in K$ whenever $g(x)$, $h(x)\in K$, then $f(x)\in K^{**}$; hence suppose that $g(x)$, $h(x)\in K$. Let $\epsilon>0$. Since $x^2f(x)\in K$, there exists a constant $M_1>0$ such that $xf(x)<M_1$. Since $g(x)$, $h(x)\in K$, there exist constants $M_2>0$ and $M_3>0$ such that $g(x)/x<M_2$ and $h(x)/x<M_3$. Moreover there exists a $\delta_1>0$ such that

$$|\,g(x)h(x)/x - g(y)h(y)/y\,| < \epsilon/(3M_1)$$

when $|x-y|<\delta_1$, since $g(x)\in K$, $h(x)/x\in K^*$. By Lemma 2.3, there exists a $\delta_2>0$ such that $y^2|f(x)-f(y)|<\epsilon/(3M_2M_3)$ when $|x-y|<\delta_2$. Take $\delta<\text{Min}\,(1,\ \delta_1,\ \delta_2,\ \epsilon/(9M_1M_2M_3))$. Then

$$\begin{aligned}
&|f(x)g(x)h(x) - f(y)g(y)h(y)| \leqq xf(x)\,|\,g(x)h(x)/x - g(y)h(y)/y\,| \\
&\quad + xf(x)\,|\,g(y)h(y)/y - g(y)h(y)/x\,| + h(y)g(y)\,|f(x)-f(y)| \\
&< M_1\cdot\epsilon/(3M_1) + M_1(g(y)/y)(h(y)/y)(y/x)\,|x-y| + (g(y)/y)(h(y)/y)y^2\,|f(x)-f(y)| \\
&< \epsilon/3 + M_1M_2M_3 3\cdot\epsilon/(9M_1M_2M_3) + M_2M_3\cdot\epsilon/(3M_2M_3) = \epsilon
\end{aligned}$$

when $|x-y|<\delta$. Hence $f(x)g(x)h(x)\in K$ and $f(x)\in K^{**}$.

Further properties of the classes K^* and K^{**} are noted in the following obvious theorems.

THEOREM 3.3. *If $f(x)$, $g(x)\in K^*$, then $f(x)+g(x)$ and $f(x)g(x)\in K^*$.*

THEOREM 3.4. *If $f(x)$, $g(x)\in K^{**}$, then $f(x)+g(x)$ and $f(x)g(x)\in K^{**}$.*

5. The definitions of K, K^*, K^{**} naturally suggest a definition of $K^{**\cdots *}$ by induction and it might be of interest to study properties of these extended classes. In addition there is still the question of necessary and sufficient conditions for $f(x)g(x)$ to be in K whenever $f(x)$, $g(x)$ are in K. In connection with this it can be established that $f(x)\in K$ whenever $f^2(x)\in K$ and $f(x)$ is continuous on $[1,\ \infty)$.

CLASSROOM NOTES

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BOOLEAN ALGEBRA

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Rosenbloom [1] suggests that the factors of 210 form a Boolean algebra if one defines $A \cup B$ as the least common multiple of A and B, and $A \cap B$ as the greatest common divisor of A and B. Andree [2] has a problem indicating the same result for the number 30. The complement of x is $210/x$ and $30/x$, respectively.

The results can be easily generalized. Let $p_1, \cdots, p_n$ be a collection of distinct primes, and consider the set theory algebra of the 2^n subsets. Now associate to each set the product of the numbers in the set (to the null set is associated unity) and consider the algebra which is formed according to the definitions in the first paragraph. The two algebras are isomorphic.

The result is being considered in the analysis of switching circuits.

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MATRIC GROUPS

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There is a commonly-held fallacy that a set of matrices which forms a group under matric multiplication must contain only nonsingular matrices.

The assignment

$$a \to \begin{bmatrix} 0 & 0 \\ -a & a \end{bmatrix} = f(a)$$

establishes an isomorphism between the real field and a subset of the set of all two-by-two matrices, using the usual matric operations. Excluding $a = 0$ one obtains a multiplicative group, even though *each* matrix is singular.

The difficulty is that "inversion" has two different meanings. For $a \neq 0$ the matric-inverse of $f(a)$ does not exist while $f(a)$ has the obvious inverse $f(a^{-1})$ relative to the identity element $f(1)$.

The astute undergraduate may observe that the single element $f(1)$ is a group under multiplication and that—even worse!—the single element $f(0)$ is a group.

STRONG CONTINUITY IN TOPOLOGICAL SPACES

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A well known necessary and sufficient condition for $f: S \to T$ to be continuous is that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset S$. We attempt in this note to investigate the strength of the more restrictive condition $f(\overline{A}) \subset f(A)$ for all $A \subset S$. When S is locally connected we can obtain a characterization of such functions.

DEFINITION 1. *A single valued function $f: S \to T$ will be termed strongly continuous (or called an S.C. function) if and only if $f(\overline{A}) \subset f(A)$ for all $A \subset S$.*

It is quite clear that $f: S \to T$ is *S.C.* if and only if $f(A') \subset f(A)$ for all $A \subset S$ where A' denotes the derived set of A.

THEOREM 1. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is closed for all $B \subset T$.*

Proof. Necessity. Let $p \in (f^{-1}(B))'$. Then $f(p) \in ff^{-1}(B)$ and $ff^{-1}(B) \subset B$ and hence $f(p) \in B$. Thus $p \in f^{-1}(B)$ and $f^{-1}(B)$ is closed. Sufficiency. Let $A \subset S$. Then $A \subset f^{-1}f(A)$ and $A' \subset (f^{-1}f(A))' \subset f^{-1}f(A)$ since $f^{-1}f(A)$ is closed. Thus $f(A') \subset f(A)$ and $f: S \to T$ is *S.C.*

COROLLARY 1. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is open for all $B \subset T$.*

The reader will easily construct a proof using complements.

COROLLARY 2. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is both open and closed for all $B \subset T$.*

The proof follows immediately from Theorem 1 and Corollary 1.

THEOREM 2. *Let $f: S \to T$ be S.C. and A a nonempty connected subset of S. Then $f(A)$ is a single point.*

Proof. Suppose $f(A)$ contains more than one point. Let $p \in f(A)$. Then $f^{-1}(p) \cap A$ is a proper subset of A and by Corollary 2 is both open and closed in A. Thus A is not connected, a contradiction.

The converse of Theorem 2 is false. For, suppose S: 0, 1, $\frac{1}{2}$, $\cdots$, $1/n$, $\cdots$ with the usual topology on the line. Let T be the space of the reals. Define $: S \to T$ as follows: $f(1/n) = 0$ and $f(0) = 1$. Then $f: S \to T$ does take nonempty connected sets in S into points, but it is clearly not continuous, and hence not *S.C.*

We have, however, a partial converse to Theorem 2 in the following theorem.

THEOREM 3. *Let $f: S \to T$ be a single valued transformation and S locally connected. Let $f(B)$ be a single point whenever B is a connected nonempty subset of S. Then $f: S \to T$ is S.C.*

Proof. We will show that $f(A') \subset f(A)$. Let $p \in A'$. Since S is locally connected, we have an open connected set 0 containing p and thus $0 \cap A \neq \phi$. Now $f(p) \in f(0)$ and $f(0)$ is a single point. Also $\phi \neq f(0 \cap A) \subset f(0) = f(p)$. Hence $f(p) = f(0 \cap A)$ and thus $f(p) \in f(A)$.

EXCEPTIONAL EXTREMUM PROBLEMS

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The elementary calculus method of locating the maximum or minimum points of a function of one variable may fail for either of two reasons. (1) The function is stationary outside the range applicable to the problem, resulting in the well known "end point" maximum or minimum. This situation is usually, but not always, easily recognizable [1]. (2) The failure is caused by the use of what Widder calls an unsuitable independent variable [2]. It is often difficult to predict when this will happen.

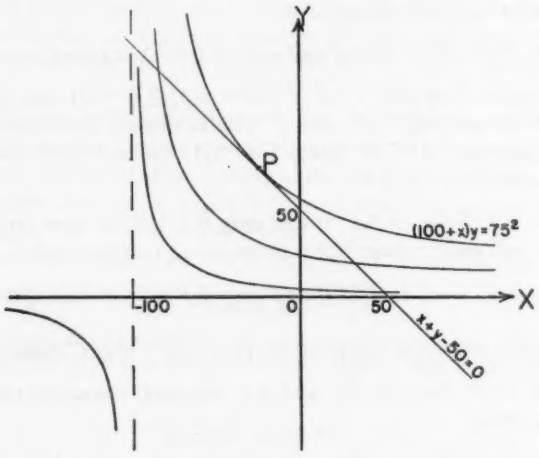

FIG. 1

The method of Lagrange multipliers may or may not get around the difficulty. In the case of an end point maximum or minimum it will frequently be of no help. Consider the following problem. Given a straight fence 100 feet long, we wish by adding 200 feet more to form a rectangular enclosure whose boundary contains all of the original fence. How shall this be done so as to enclose the greatest possible area? [3] If we let x be the length of the new fence which is to be aligned with the original 100 feet and proceed in the usual fashion to maximize the resulting expression for area, x comes out to be -25, which is not permitted by the conditions of the problem. If we use a Lagrange multiplier to maximize the function $f(x, y)=(100+x)y$ subject to the restraint $g(x, y)=100+2x+2y-200=0$, we get the same result. The method of Lagrange must lead to a point or points where one of the curves $f(x, y)=c$ is tangent to $g(x, y)=0$ [4]. Figure 1 indicates what is happening. A member of the family of hyperbolas $(100+x)y=c$ is tangent to the line $x+y-50=0$ at the point P: $(-25, 75)$, the

correct values of (x, y) for absolute maximum area, a square enclosure 75 feet on a side. But in order to bring the solution within the stipulation of the original problem we need $x=0$, an end point maximum. The function is not stationary there, of course; and from this point of view the failure of both the elementary and the Lagrange method is essentially trivial. They could not be expected to succeed.

Consider next Widder's example: to find the shortest distance from the focus (1, 0) to the parabola $y^2=4x$. Here $f(x, y)=(x-1)^2+y^2$ must be minimized subject to the restraint $g(x, y)=y^2-4x=0$. An attempt to eliminate y from $f(x, y)$ and set $f'=0$ yields $x=-1$, a point not on the parabola. Of course the method of Lagrange succeeds, because the circle $f(x, y)=c^2$, c being the distance from the focus, can become tangent to the parabola at the required point (0, 0). We have $u=f(x, y)+\lambda g(x, y)=(x-1)^2+y^2+\lambda(y^2-4x)$,

$$\partial u/\partial x = 2(x-1)-4\lambda = 0,$$
$$\partial u/\partial y = 2y+2\lambda y = 0,$$
$$\partial u/\partial \lambda = y^2-4x = 0.$$

In the simultaneous solution of these three equations in this particular example one "could not eliminate λ by solving the second equation thereof for λ. For, $\partial g/\partial y=0$ at the very point which yields the minimum" [2]. But actually, the implication that the two conditions in quotations are generally interdependent is unjustified. We shall see that inability to eliminate λ in this way is not a consequence of the vanishing of $\partial g/\partial y$.

What we should like, ideally, is a criterion for judging in advance the "suitability" of an independent variable. Unfortunately this seems rather too much to ask. We might surmise, from the above example, that the difficulty occurs because x and the distance function, say s, behave in the same way at the minimum point. That is, ds/dx becomes meaningless at (0, 0) because, as one moves along the parabola, x and s become stationary at the same time. But this circumstance is in fact neither necessary nor sufficient. If one seeks the extrema of the distance from the point (2, 0) to the hypocycloid whose equation is $x^{2/3}+y^{2/3}=1$, the attempt to maximize s as a function of x fails to yield the local maxima at $(0, \pm 1)$ even though s and x are not "behaving similarly" at these points. (Later we shall find the converse counter example.) This is also an illuminating illustration of the limitations of Lagrange multipliers, which yield only the local maxima at P_1, P_2, where the circle representing the distance function is tangent to the curve, and none of the cusps (Fig. 2).

We digress temporarily to investigate the apparently peculiar state of affairs near the focus in Widder's problem. In order to simplify the details of the necessary algebra, we consider instead a similar example: [5] to find that point on the circle $x^2+y^2=1$ nearest to $(\frac{1}{2}, 0)$. If we let $L^2=s$, the square of the distance from $(\frac{1}{2}, 0)$ to any point (x, y) on the circle, we have $s=(\frac{1}{2}-x)^2+1-x^2$, and setting $ds/dx=0$ yields the absurd result $-1=0$. To be slightly more general,

we can say that if s is the squared distance from a point on the circle to any point (a, b), $a \neq 0$, the elementary method fails when $b=0$. However, for any $b \neq 0$ but *arbitrarily close to zero*, the same method is successful. For instance, if P is a point which is allowed to assume various positions along the line $x=a \neq 0$, then for every position except $(a, 0)$ there is no trouble. What causes this sudden discontinuity?

The expression for s is

$$s = (x-a)^2 + (\pm\sqrt{(1-x^2)} - b)^2. \tag{1}$$

If $b=0$

$$s = a^2 + 1 - 2ax, \tag{2}$$

a straight line of slope $-2a$ in the (x, s)-plane. Admissible values of x lie in the interval $-1 \leqq x \leqq 1$, and if x equals 1 or -1, $L = |a-1|$ or $|a+1|$, the end point minimum or maximum.

If $b \neq 0$, the curve of (1) is an ellipse. For any fixed a (we take $a=\frac{1}{2}$ in Fig. 3), we have a family of ellipses with b as the parameter, whose envelope is the pair of parallel lines $x = \pm 1$. Each ellipse is tangent to both lines and hence is centered on the s-axis; its maximum and minimum points are the maximum and minimum s for that particular a and b. The major axis of each ellipse is inclined to the horizontal at an angle ϕ whose tangent is negative and numerically larger than $2a$. As b tends toward ∞ or $-\infty$ for any fixed a, $\phi \to \frac{1}{2}\pi$ and the ellipse moves upward, both s(max) and s(min) tending toward $+\infty$. As $b \to 0$, $\tan\phi \to -2a$. The segment of the straight line of (2) lying in the interval $-1 \leqq x \leqq 1$ is a degenerate ellipse, the limiting position of the ellipses of the family. If we differentiate (1) with respect to x and set $ds/dx = 0$, and then eliminate the parameter b between this equation and (1), we obtain the locus of the maximum and minimum of s, namely

$$s = (1 - a/x)^2. \tag{3}$$

As $b \to 0$, the ordinary maximum and minimum points of the nondegenerate ellipses approach the end point maximum and minimum points of the degenerate ellipse continuously along the two branches of the curve of (3), thus accounting for the apparent discontinuity mentioned at the outset. This kind of behavior is typically encountered in such problems.

We turn now to our final example: to find the cone of maximum lateral surface area with vertex at the origin and inscribed in the ellipsoid formed by rotating the curve $y^2/a^2 + x^2/b^2 = 1$ around the Y-axis (Fig. 4). If one expresses the lateral surface area S as a function of x and sets $dS/dx = 0$, the result yields the true maximum if $a^2 > 2b^2$. If $a^2 = 2b^2$, the maximum occurs at the point $(b, 0)$ where the cone has flattened out into a circular disc. The elementary method yields this result even though x and S are stationary at the same time, the converse counter example mentioned earlier. If $a^2 < 2b^2$, the maximum is again at

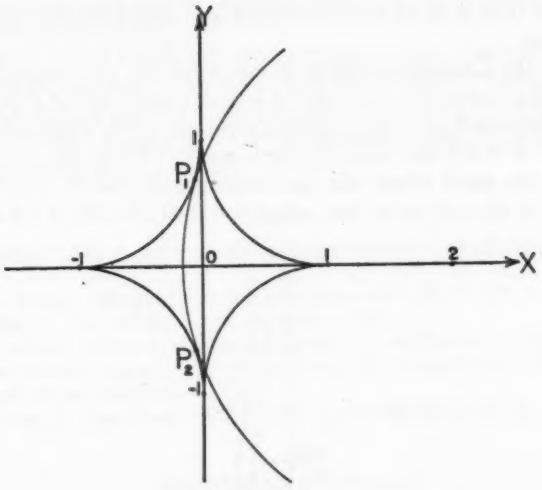

FIG. 2

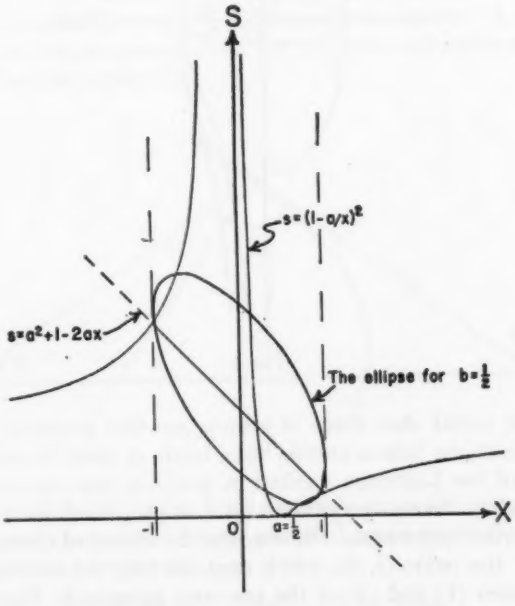

FIG. 3

$(b, 0)$, but this time it is an end point case and hence not revealed by the elementary attack.

If one uses the Lagrange method to maximize $f(x, y)=\pi x\sqrt{(x^2+y^2)}$ subject to the restraint $g(x, y)=y^2/a^2+x^2/b^2-1=0$, one finds: (1) If $a^2>2b^2$, the maximum cone is delivered and also a second solution, $y=0$, overlooked by the first method. (2) If $a^2=2b^2$, the circle is the answer. This is the case where $\partial g/\partial y=2y/a^2=0$ at the point where the maximum occurs, yet it is perfectly feasible to eliminate λ in the course of the solution. (3) If $a^2<2b^2$, $y=0$ as in (1).

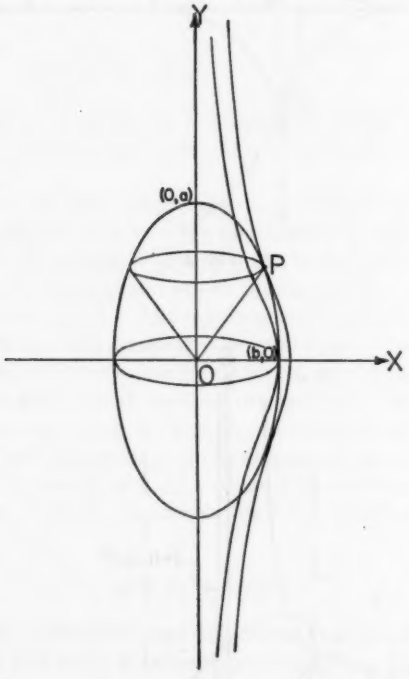

FIG. 4

It should be noted that there is always another extremal at $(0, a)$, the limiting case where the lateral surface area tends to zero. As usual, the reason for the failure of the Lagrange method to produce this minimum is that the curves $f(x, y)=$constant can never be tangent to $g(x, y)=0$ there; they have the line $x=0$ as a vertical asymptote. On the other hand, one of them can be tangent to $g(x, y)=0$ at the point $(b, 0)$, which explains why the method picks up the end points in cases (1) and (3) of the previous paragraph. Figure 4 shows an ellipse for which $a^2>2b^2$, and two tangent curves $f(x, y)=c$. The lateral surface

area of the inscribed cone has its absolute maximum at the point P and a local minimum at the point $(b, 0)$.

There exists a device which will yield *all* the extrema: a suitable parametrization of the original variables. This is not difficult to effect. The method of choosing the parametrization has been indicated by Oakley [6]. The interested reader can find all six of the extrema in the hypocycloid example by the use of the conventional parametrization, $x = \cos^3\theta$, $y = \sin^3\theta$.

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4. See, for instance, Angus E. Taylor, Advanced Calculus, Boston, 1955, Sec. 6.8.
5. Frank Hawthorne, A simple endpoint maximum, this MONTHLY, vol. 58, 1951, p. 188. (The problem is stated but not analyzed.)
6. C. O. Oakley, Endpoint maxima and minima, this MONTHLY, vol. 54, 1947, pp. 407–409.

CURVATURES OF $r^n = \cos n\theta$

ROBERT C. YATES, College of William and Mary

Familiar members of the family of curves with polar equation $r^n = \cos n\theta$ are the *circle* for $n = 1$, the *lemniscate* for $n = 2$, the *line* for $n = -1$, the *hyperbola* for $n = -2$, the *cardioid* for $n = \frac{1}{2}$, and the *parabola* for $n = -\frac{1}{2}$. For rational n, all members have simple constructions for tangents and centers of curvature if multisection of angles is permitted.

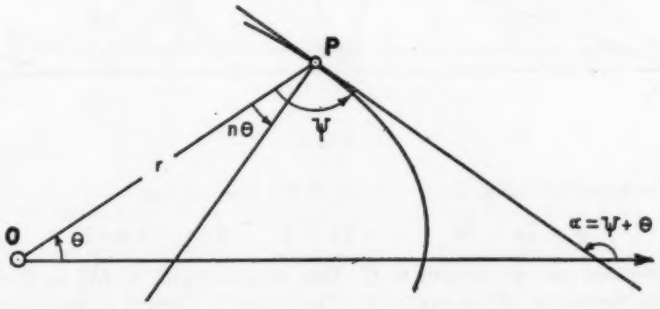

FIG. 1

Let α be the angle formed by a tangent with the polar axis, ψ the angle from OP to the tangent (counterclockwise, positive), and s an arc length of the curve (Fig. 1). Then from $r^n = \cos n\theta$,

$$r^{n-1}r' = -\sin n\theta, \qquad [' \equiv d/d\theta] \tag{1}$$

and

$$\tan \psi = r/r' = -\cot n\theta = \tan (\tfrac{1}{2}\pi + n\theta).$$

Accordingly,*

$$\psi = \tfrac{1}{2}\pi + n\theta \quad \text{and} \quad \alpha = \psi + \theta = \tfrac{1}{2}\pi + (n+1)\theta.$$

Thus the normal at P forms the angle $n\theta$ with OP. From (1) the sum of the squares gives

$$r^{2n-2}(r^2 + r'^2) = 1 \quad \text{or} \quad ds/d\theta = 1/r^{n-1}.$$

The radius of curvature at P is thus

$$R = ds/d\alpha = (ds/d\theta)(d\theta/d\alpha) = (1/r^{n-1})[1/(n+1)],$$

or, since $r^n = \cos n\theta$,

$$R = \{r/(n+1)\} \cdot \sec n\theta.$$

1. *The lemniscate* (Fig. 2): $r^2 = \cos 2\theta$ for $n = 2$ has

$$\psi = \tfrac{1}{2}\pi + 2\theta, \qquad \alpha = \tfrac{1}{2}\pi + 3\theta, \qquad R = \tfrac{1}{3}r \sec 2\theta.$$

The length $\overline{OP} = r$ is trisected to give $\overline{QP} = \frac{1}{3}r$. The perpendicular to OP at Q meets the normal at P in the center of curvature C.

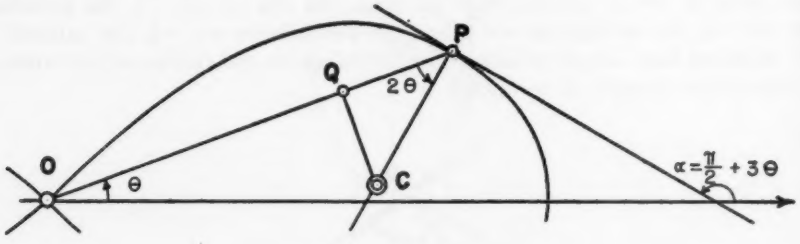

FIG. 2

2. *The hyperbola* (Fig. 3): $1 = r^2 \cos 2\theta$ for $n = -2$ has

$$\psi = \tfrac{1}{2}\pi - 2\theta, \qquad \alpha = \tfrac{1}{2}\pi - \theta, \qquad R = -r \sec 2\theta.$$

OP is extended its own length to Q. The perpendicular to OQ at Q meets the normal in the center of curvature C. Note that the length of the normal from P to the polar axis is r.

3. *The cardioid.* (Fig. 4): $r^{1/2} = \cos \frac{1}{2}\theta$ (or $2r = 1 + \cos\theta$) for $n = \frac{1}{2}$ has

$$\psi = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta, \qquad \alpha = \tfrac{1}{2}\pi + \tfrac{3}{2}\theta, \qquad R = \tfrac{2}{3}\cos \tfrac{1}{2}\theta.$$

* Actually $\psi = \frac{1}{2}\pi + n\theta + k\pi$, k integral. Numbers k other than zero lead to the same tangent line.

The projection of the constant length $\overline{PQ}=\frac{3}{2}$ on the normal is the radius of curvature and locates the center C.

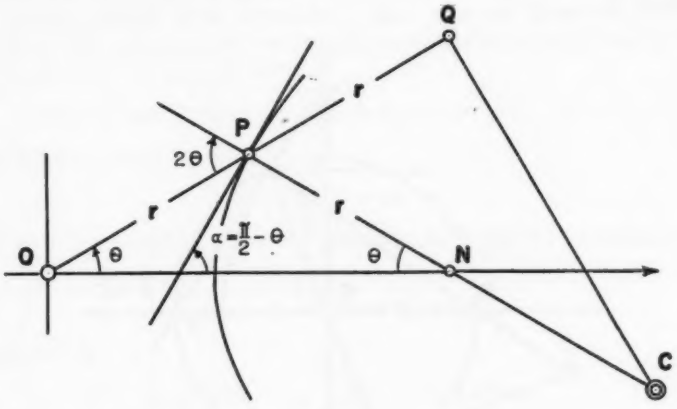

FIG. 3

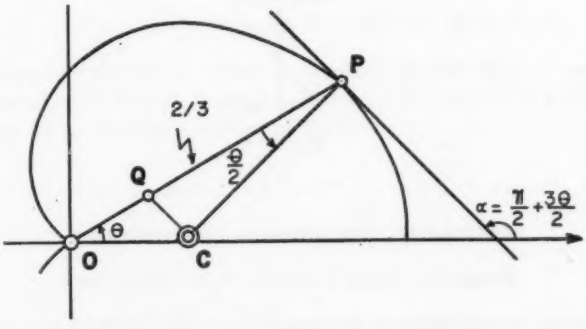

FIG. 4

4. *The parabola.* (Fig. 5): $r^{-1/2}=\cos\frac{1}{2}\theta$ [or $r(1+\cos\theta)=2$] for $n=-\frac{1}{2}$ has

$$\psi=\tfrac{1}{2}\pi-\tfrac{1}{2}\theta,\qquad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\theta,\qquad R=2r\sec\tfrac{1}{2}\theta.$$

PO is extended to Q with $\overline{OQ}=r$. The perpendicular to QP at Q meets the normal in the center of curvature C. Note, since $\alpha=\frac{1}{2}(\pi+\theta)$, that the inclination of the normal is $\frac{1}{2}\theta$. The reflective property with point source at the focus 0 is thus established. Note further since $\overline{ON}=r$ that $\overline{NP}=2r\cos\frac{1}{2}\theta$. Thus the projection of this normal length on the polar axis (a subnormal) is the constant:

$$\overline{PN}\cos\tfrac{1}{2}\theta=2r\cos^2\tfrac{1}{2}\theta=2.$$

Construction is simple: with center at the focus, draw a circle meeting the

parabola in P, the polar axis in N, T, and the extension of PO in Q. The tangent and normal at P are PT and PN. The center of curvature lies on the tangent to the circle at Q.

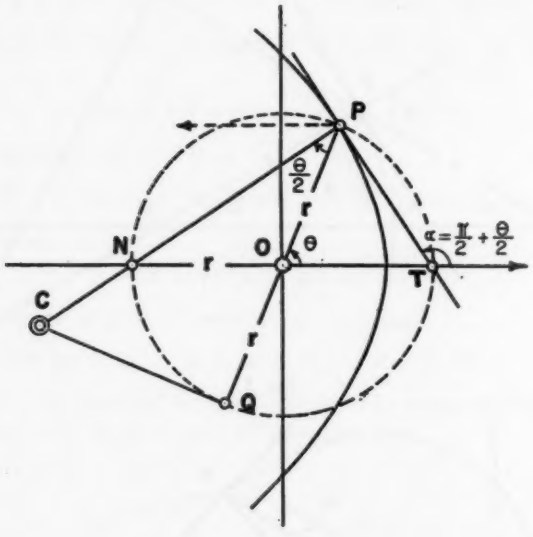

FIG. 5

MOTIVATING THE METHOD OF FROBENIUS

ROBERT H. OWENS, University of New Hampshire

The following note is believed to complement the interesting article by R. D. Larsson in the August-September 1958 issue of this MONTHLY.

The student is easily convinced that general solutions of linear differential equations may be obtained in the form of power series about an ordinary point at $x=0$ by introducing

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

into the differential equation.

When $x=0$ is a regular singular point he is told to introduce

$$y = x^a \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

because (1) may not work. If sufficiently curious he may even try (1). This "device" (2) can be justified, of course, by a theoretical discussion of the method of Frobenius. However, this is not always done, especially in "methods" courses.

The matter can be more successfully dealt with by observing that every linear differential equation (of the second order for simplicity) with a regular singular point at $x=0$ may be written

$$x^2y'' + x(p_0 + p_1x + \cdots)y' + (q_0 + q_1x + \cdots)y = 0. \tag{3}$$

For x sufficiently small we have

$$x^2y'' + p_0xy' + q_0y \approx 0$$

which is an Euler equation whose solution may be found by introducing $y=x^m$ so that

$$m^2 + (p_0 - 1)m + q_0 = 0. \tag{4}$$

Then, near $x=0$

$$y_1 \approx x^{m_1}, \qquad y_2 \approx x^{m_2} \qquad \text{if } m_1 \not\approx m_2 \tag{5a}$$

and, as in Larsson's note,

$$y_1 \approx x^{m_1}, \qquad y_2 \approx x^{m_1}\log x \equiv y_1 \log x \qquad \text{if } m_1 = m_2. \tag{5b}$$

Thus, the introduction of (1), which in general implies that $y \approx a_0$ near $x=0$, cannot be expected to lead to a series solution of (3). And it is a natural consequence of (5a, b) to assume solutions of the form

$$y_1 = x^{m_1}\sum_{n=0}^{\infty} a_nx^n, \qquad y_2 = x^{m_2}\sum_{n=0}^{\infty} b_nx^n, \qquad m_1 \not\approx m_2 \tag{6}$$

or

$$\begin{aligned} y_1 &= x^{m_1}\sum_{n=0}^{\infty} a_nx^n, \\ y_2 &= y_1[\log x + c_0 + c_1x + \cdots] \\ &\equiv y_1 \log x + x^{m_1}\sum_{n=0}^{\infty} b_nx^n, \qquad m_1 = m_2. \end{aligned}$$

Incidentally, (4) is the indicial equation for (3) and can always be obtained in this manner, i.e. by grouping the terms which are dimensionally homogeneous in x^m of lowest degree and obtaining the characteristic equation of this expression as if it were an Euler differential equation.

In the event that m_1, m_2 differ by an integer it is well known that (6) will still lead to one solution corresponding to the larger of m_1 and m_2, the second solution being obtained by reduction of order. However, one usually tries the smaller of m_1, m_2 first since both solutions (or none) may result.

A PEDAGOGICAL ASPECT OF THE DEVELOPMENT OF THE REAL NUMBERS

ROBERT S. LEDLEY, The George Washington University

Courses in real variables frequently include some set theory followed by a discussion of infinite sets and the associated cardinals and ordinals. Such courses usually also include as a separate topic the classical development of the real numbers starting from Peano's axioms for the positive integers (see, for example, the text by Thielman [1]). There are two disadvantages to such separate treatments of these two topics: first, it imposes a break in the continuity of the mathematical development of the course; and second, Peano's axioms themselves appear rather arbitrary and perhaps unsatisfying, especially because finite induction reads more like a theorem than a postulate. The purpose of this brief note is to suggest what is in the author's opinion a pedagogically and philosophically more satisfying unified approach to this aspect of the development of the real numbers. This approach would be first to consider infinite sets and ordinals and cardinals; then the *positive integers* can be defined as an *infinite well-ordered set of which every element except the first has an immediate predecessor.* Since the student will have already been introduced to the infinite and to well-ordering through the study of the cardinals and ordinals, such a definition would more naturally appeal to the basic intuitive idea of the positive integers as well as unify the topics of cardinals, ordinals, and positive integers. Finite induction may then be proved as a theorem, and the real numbers then developed from the positive integers in the conventional manner with no other changes.

One procedure might be first to prove the so-called principle of transfinite induction, and from this to prove Peano's finite induction postulate. A standard proof proceeds as follows:

THEOREM (*principle of transfinite induction*). *Let A be a well-ordered set, and let $B \subset A$ such that*

(1) *B includes the first element of A, and*

(2) *If $a \in A$ implies that all elements preceding a are in B, then $a \in B$.*

Then we can conclude that $A = B$.

Proof. Suppose $A \neq B$. Then let $C \subset A$ ($C \neq \phi$) be all elements of A not in B. Then C does not contain the first element of A [because B does by hypothesis (1)]. Hence there exist elements of B preceding all elements of C. Let $c_0 \in C$ be the first element of C. Then all elements preceding c_0 are in B. Then, by hypothesis (2), $c_0 \in B$ contradicting the construction of C. Hence $A = B$. Q.E.D.

The advantage of this proof is that it has only made use of the properties of well-ordered sets, with which the student should now be familiar. Next observe that every element of a well-ordered set A has an immediate successor (*e.g.*, for a' to be the immediate successor of a, we can define a' as the first element of the set $S = \{a^* | a^* \in A,\ a^* > a\}$. Also, if the immediate predecessor a of a' exists, then a' is the immediate successor of a.

Now we are ready to prove finite induction, based on our definition of the

positive integers given above.

THEOREM (*Peano's postulate of finite induction*). *Let A be the set of positive integers, and let $B \subset A$ such that*

(1) *B includes the first element of A, and*

(2) *$a \in B$ implies that its immediate successor $a' \in B$.*

Then we can conclude that $A = B$.

Proof. Hypothesis (1) of transfinite induction is the same as hypothesis (1) of finite induction. Next suppose that $a' \in A$ implies that all elements preceding a' are in B. Then, since the immediate predecessor of a' is in B by hypothesis (2), $a' \in B$; thus hypothesis (2) of transfinite induction is satisfied. Thus by the transfinite induction theorem, $A = B$.

The rest of Peano's axioms follow almost trivially, and the development of the real numbers can now proceed in the classical form.

Reference

1. Henry P. Thielman, Theory of Functions of Real Variables, Englewood Cliffs, N. J., 1953, Chs. I and II, pp. 1–51.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE ROLE OF GEOMETRY FOR THE MATHEMATICS STUDENT*

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The importance of geometry in college mathematics has declined for some time, in fact there are now quite a few colleges which offer no course at all in geometry proper. While this development does not worry all mathematicians, it caused concern to sufficiently many to start the present discussion. Here we first deal with the present status of geometry in college, *i.e.*, with the courses which are actually offered, then try to evaluate these courses in view of the aims of geometry teaching, and will arrive at some definite proposals regarding courses to be eliminated or added. The aims, and consequently the evaluation and the proposals, necessarily depend partly on the speaker's views and taste.

The presentation of the actual situation is principally based on the information furnished by the catalogues of 182 colleges, including all the better known ones. The catalogues usually fail to provide all the relevant facts. Course de-

* Presented at the Fortieth Summer Meeting of the Association, Salt Lake City, September 1, 1959.

scriptions are often inadequate or absent. For example, it would have been important for our present purposes to know whether a course in projective geometry uses an algebraic, a geometric or a mixed approach. There is often no indication, which may have the very good reason that the choice of method is left to the teacher. Also, the frequency with which a course is given is not always listed.

Nevertheless the following two tables extracted from the catalogues are believed to be useful; they give the total number of courses in geometry per college and for each individual course the number of colleges teaching it. However it must first be stated which courses are taken into account. Since this lecture is concerned with undergraduate teaching only, graduate courses in geometry do not figure in the tables unless they are explicitly listed as open to undergraduates. This point principally concerns differential geometry. Vector analysis is not counted although it almost always comprises some solid and some differential geometry. Above all, the standard course: "Analytic Geometry and Calculus" is disregarded. This course originated from the waning interest in geometry, almost always proceeds at the expense of geometry,* and is largely responsible for perpetuating the condition.

TABLE 1

Total number of geometry courses per college

Number n of courses in college	Number of colleges offering n courses
0	13
1	23
2	75
3	39
4	24
5	4
6	4

TABLE 2

Types of Geometry Courses

Course		Number of colleges offering this course
Analytic Geometry		58
Projective Geometry		
undefined	69	
algebraic	20	99
geometric-axiomatic	10	
Foundations of Geometry		9
Noneuclidean Geometry		29
Higher Geometry		33
Modern Euclidean Geometry (triangle and circle)		52
Differential Geometry		61
Other		21

The sum of the numbers in the right column does not equal $1\times23+2\times75+\cdots+6\times4$ because two semester courses in the same subject, like Higher Geometry I, II, appear only once on the right and twice on the left.

Examining the individual courses, we notice that Differential Geometry is in a different position from the rest. Its importance is generally admitted, the subject flourishes. But this leads naturally to the tendency of introducing the student to the now active parts of the field. The course then becomes an advanced graduate course. On the other hand the theory of surfaces in E^3 is a very rich field still quite active, and many of its results are not contained in the

* Since this talk was given Angus E. Taylor's Calculus with Analytic Geometry appeared. It contains most of the material taught in analytic geometry.

type of advanced course just mentioned. Perhaps every undergraduate ought to have a chance (but should not be required) to learn the classical theory, which is also useful in many applications.

There are influential mathematicians doubting the value of any of the remaining courses. We will discuss them from three points of view which may be briefly listed as: factual information, axiomatic thinking, training of the geometric intuition.

Nobody denies that some analytic geometry is necessary, but geometers doubt that "Analytic Geometry and Calculus" suffices. Nevertheless, since saving time for newer developments is essential, this situation may be accepted, provided the student becomes familiar with projective coordinates. Our table shows that projective geometry is by far the most popular course in geometry. The usual objection to the course (which also turned up in the discussion after this lecture) is that the topic is no longer an active subject of research; and that it can easily be introduced (but not adequately treated) as an appendix to the now popular vector spaces. The principles underlying this attitude are highly dangerous. The subject is beautiful, it required major efforts of some of the best mathematicians during and after the last century to understand its structure thoroughly, it still gives a deep insight into geometry. The projective duality principle, besides being constantly used by geometers, is an exciting experience for any interested student. The duality in vector spaces is a very meager substitute. But most of all, do we have the right to completely disrupt historical continuity whenever a subject moves out of the focus of contemporary interests? Do we really expect or agree that our present mathematical efforts will be altogether junked (at least from courses) as soon as the interests change? This disregard for historical continuity obviously leads to dangerous absurdities if taken seriously and must be fought. As far as geometry is concerned the speaker believes that projective geometry is a subject on which to make a stand. A mathematics major should not obtain his degree without knowing projective geometry. There remains the question how the subject should be taught. It will be seen later why the axiomatic approach seems less important now, hence a course should preferably contain both algebraic and geometric arguments. This freedom in method also has the advantage of yielding results most quickly.

Noneuclidean geometry played, historically, the important role of an eye opener. It will still have this effect on the student, moreover it lends itself readily for an exciting undergraduate course and should therefore be offered once in a while either by itself or as a part of another course.

The value of higher geometry as a course depends on its content. It often consists mainly of projective and noneuclidean geometry; then the preceding arguments justify its existence. All too often the content is left to the instructor. Instances are not rare, where the course is assigned to a nongeometer, and then degenerates, for example, into the pure algebra connected with constructions by ruler and compass.

Modern euclidean geometry appears under various synonyms, for example

"Modern (or Advanced) Geometry of Circles and Triangles." It deals with those facts of elementary euclidean geometry which were discovered after Apollonius, so that the definition "modern" used here means "not more than 2100 years old." The course consists of individual niceties entirely unrelated to anything else. It has a slight merit as a training of the intuition and for high school teachers, but we will see later that this can be accomplished far better. The course should therefore disappear.

We now come to our second point, namely axiomatics and in particular to the foundations of geometry as a course, which was passed over above. Not so very long ago this course and axiomatic projective geometry constituted (for historical reasons) the only opportunity for introducing the student to axiomatic thinking, which greatly appeals to many minds with a mathematically critical or a logical bent. Now large parts of mathematics are treated axiomatically, and, what is worse, the axioms for a group, for a field, or for a Hausdorff space, are simpler by far than those for euclidean geometry. Therefore the foundations of geometry no longer satisfy a real need; the same applies to the axiomatic approach to noneuclidean or projective geometry, although the axioms for the latter are simple. Many geometers will find it hard to admit this, because axiomatics is very close to their hearts and is still a rather active field of research. Students interested in the subject turn up now and then, but they can very well be helped by a reading course. In this connection it should be considered whether the modern development of the foundations of geometry on the basis of group theory* might not be preferable to Hilbert's approach.

The shift in mathematics towards the axiomatic approach and the concomitant abstractness make it imperative that some course in geometry strengthen the geometric intuition which is in great danger of being lost, but has always been one of the main inspirations in mathematical creation. The abstract approach not only fails to develop the intuition, but the unexpected phenomena arising in very general situations often completely destroy the confidence in intuitive arguments just in the best students. When "moving a hyperplane in E^n continuously" (in the course of a proof), the author was repeatedly interrupted with the question: "Which topology for hyperplanes do you use?" Needless to say that a student who has qualms regarding this operation is forever lost to geometry and geometric reasoning.

The course in "Modern Euclidean Geometry" does train the intuition to a certain extent, but is objectionable for the reasons already given and, in the present context, because it seems to confirm the rather prevalent feeling that modern mathematics is too advanced or intricate for the intuition.

A glance at *Mathematical Reviews* shows that the theory of convex bodies is quite active. It is applied in many other fields, for example in game theory; its proofs are often purely synthetic. Didactically, convexity has the advantage

* A book on this topic is: F. Bachmann, Aufbau der Geometrie aus dem Spiegelungsbegriff, Berlin, 1959.

over the classical subjects of euclidean geometry that it is defined essentially by an inequality and hence requires much deeper arguments. The proof that convexity can be treated in a form accessible to sophomores is provided by the book of I. M. Yaglom and V. G. Boltyanski on convex figures, which was first published in 1951 in Russia, has since appeared in a second Russian edition and in a German translation (*Konvexe Figuren*, Verlag der deutschen Wissenschaften, 1956), and will soon appear in an English translation (P. J. Kelly and L. F. Walton). It deals with plane convex figures, but leads there to very subtle topics, providing training in continuity arguments, applications of Helly's Theorem, addition of convex figures, maximum and minimum problems with and without solutions, curves of constant width and their generalizations.

The material is divided into theorems or lemmas posed as problems occupying one third, and the solutions occupying the remaining two thirds, of the book, but it can be used as a text.

This book was discussed at some length because there is at present no other book satisfying our requirements, namely, to make strong demands on the intuition, to furnish a useful training for high school teachers as well as to those contemplating research, and to be accessible to sophomores, say. Thus the book constitutes a constructive existence proof, but we do not contend that the problem has only one solution. It is up to the geometers here and elsewhere to discover other possibilities. The more we try the better the chance will be of arriving at a synthesis of ideas which will reverse the present adverse trend in geometry.

TEACHING MACHINES AND LOGIC

JOHN W. BLYTH, Hamilton College

The nearly universal applicability of mathematics and logic stems in large part from their abstract and formal character. Yet this abstractness which is the source of their utility is also the source of widespread difficulty in mastering these subjects. It seems paradoxical that the two subjects with the most thoroughly rational structure of all subjects should prove to be the least intelligible to so many students.

There is now very good reason for believing that this paradox may be eliminated. The reason is the development of instructional materials programmed for presentation by a teaching machine or "learning" machine. This mechanical tutor encourages the student to accept responsibility for his own learning.

The task of programming a lesson bears some analogy to the task of preparing a problem or a set of data for automatic processing in a digital computer. It is necessary to break up a problem into a large number of small successive steps. Processing data requires the performance of many operations such as adding, dividing, printing, etc. But before each operation a question must be answered. Two of the requirements which must be met by each question are illuminating

for our present purpose. In the first place the question must be stated in the unambiguous language of the computer so that one question is clearly discriminated from another. In the second place the question must have an answer that is clearly correct or incorrect. Questions which do not meet these conditions will bring the computer to a halt or send it "looping" around in a vicious circle.

One more illuminating parallel is a characteristic built into a computer. It is designed to double check every step in a calculation. It cannot proceed to the next item in a program until the correctness of the preceding step has been determined.

A program for processing a set of data must be prepared to meet the requirements imposed by the nature of the data and the design of the computer. Similarly a program for self-instruction must satisfy the requirements imposed by the subject matter and by the character of the human learner. The learning process must be analyzed into a large number of small steps to be taken in proper order. Each step is taken in response to a question. Each question must be so presented that the learner can easily discriminate the significant features of the situation. The question must have an answer that the student can recognize as correct or incorrect. Finally, the student must check each step before proceeding. If these conditions are not all satisfied, the learning operation stops temporarily or is at least greatly retarded.

Machines require no motivation and no reward. The learner requires both. Once a student has been motivated to start a learning task success in getting the correct answer may be sufficient reward to keep him working. There is a mounting body of experimental evidence indicating that this reward or reinforcement must follow the response immediately. Since people vary in reaction time, they will be ready to check their answers at different times. This means that the instructional process must be individualized.

The teaching machine is designed for individual use and permits each student to proceed at his own pace. The machine presents a question and provides a space for the student's answer. The correct answer is made available as soon as needed. Only after checking his response is the student able to proceed. The machine presents the steps in a thoroughly tested program in a controlled sequence.

An experiment in the use of such programmed materials in teaching logic was begun at Hamilton College in the fall of 1958 with the aid of a grant from the Fund for the Advancement of Education.* Definitive conclusions cannot yet be drawn, but the preliminary results are most encouraging.

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* With an additional grant this experiment is continuing on an enlarged scale. It now includes courses in Mathematics, Psychology, French, and German in addition to Logic. The machine used in this experiment was designed by Professors Blyth, Godcharles and Jacobson and is being manufactured by The Hamilton Research Associates, Inc., Clinton, N. Y. Each lesson of 40 questions and answers is on microfilm in a thin $2'' \times 7\frac{1}{2}''$ plastic card. The student simply inserts the card in a slot and the frames are projected on a reading screen as needed. A companion audio unit is correlated automatically with the visual unit for foreign language.

Even with the programs resulting from the first effort of Professor Jacobson and myself, it was possible to reduce class contact hours by one third, increase the amount of material covered, and simultaneously raise the level of mastery. The average grade on an hour examination on probability was raised by approximately 30 points over the corresponding examination the previous year. The need for private conferences was almost eliminated. Failure for that particular class was entirely eliminated.

The character of classroom work was radically changed. Since the students had mastered no errors through untutored practice, no time was wasted in identifying and removing such difficulties. With no need for routine checking and drill the class time could be devoted to further development and varied applications of the skills and concepts developed in the lessons.

In the summer of 1959 some of the lessons prepared for college students were tried in an experimental course in the Program of Advanced Studies for Gifted Children conducted by the Board of Cooperative Educational Services of the First Supervisory District in Oneida County. It was encouraging to find the younger students, some in the seventh grade, doing work that compared favorably with that of the college students. This suggests that good programs should produce widespread mastery of complex skills at a much earlier age than now thought possible.

Much hard work remains to be done to master this new application of two ancient skills of the expert teacher. Programming combines the Socratic method of teaching by asking questions with the Cartesian method of analyzing a problem into its simplest parts and proceeding from the simple to the complex. The use of the resulting programs in teaching machines will permit superior private tutoring on a massive scale. This will not eliminate the need for human teachers, but it will surely increase their effectiveness.

Mathematics Program of the Arizona Academy of Science

In the November 1959 issue of the MONTHLY a report was given on the mathematics parts of programs of state Academies of Science under the new National Science Foundation Academy of Science project. Since this report was submitted, an additional statement has been received from the Arizona Academy of Science. This statement is as follows:

The Arizona Academy of Science does not have a Mathematics Section, but the Southwestern Section of the Mathematics Association of America is cooperating with us in our Traveling Science Institute. We include mathematics in the program and anticipate considerable activity in this field. We shall be grateful for suggestions from the Committee on Visiting Lecturers to Secondary Schools and glad to cooperate in any way possible.

NSF Summer Institutes for Engineering Teachers

Ten of the 379 Summer Institutes for high school and college teachers being supported by the National Science Foundation in 1960 will be of direct interest to teachers of engineering, preengineering, and technical institute subjects. Information about these ten Summer Institutes appeared in the January, 1960, issue of the Journal of Engineering Education. Further information may be obtained from Mr. Clyde C. Hall, National Science Foundation, Washington, D. C.

TOPOLOGICAL IDEAS IN JUNIOR HIGH SCHOOL MATHEMATICS

R. D. ANDERSON, Louisiana State University

The experimental seventh and eighth grade text materials which have been prepared by the School Mathematics Study Group introduce a number of mathematical ideas not found in traditional texts at this level. The experimental books have a greatly increased emphasis on geometric ideas and intuition. A set-theoretic approach to geometry is adopted. Thus the point of view is that of introducing geometry and geometric thought not only for general information and background but also as a vehicle for the introduction of some other significant mathematical concepts. No attempt is made to lift tenth grade geometry as such and to transplant it. There are serious efforts to get the pupils to achieve some elementary understanding of three-dimensional space.

In the trials in the experimental centers during 1958–59, the unit (or chapter) on non-metric geometry was considered on the basis of teacher reports to be rather remarkably successful. This unit dealt with set terminology in the framework of some of those concepts of geometry which do not involve measurement. This experimental unit was the precursor of the present Chapter 4 of Volume I of *Junior High School Mathematics*. In this current chapter on non-metric geometry, there is considerable further emphasis on spatial concepts and relationships. Lines, planes, and space are regarded as sets of points. The possibilities of and properties of intersections of sets of lines or planes are considered. The separation properties of space by planes, of planes by lines, and of lines by points are emphasized. Segments (regarded as sets of points) are used to help identify some of these properties.

After triangles and angles with their interiors and exteriors (all as sets of points) are introduced, there is a brief discussion of one-to-one correspondences among sets of points, sets of lines, and sets of rays. The chapter ends with a discussion of simple closed curves—anticipating applications to circles, quadrilaterals and similar objects. The further chapters of the seventh grade experimental text which are geometrical in nature deal with measurement, parallels and perpendiculars, and the circle. These subjects can all be treated effectively with set terminology and the background of Chapter 4. As the general subject matter of these later chapters is more traditional in character, it will almost certainly be better known to most junior high school teachers.

Volume II is designed basically as an eighth grade program, but for 1959–60 is not considered as a complete text. In the geometric area it has units dealing with congruence relations, similar triangles, non-metric polyhedrons, and spatial volume and area concepts. Set terminology is used throughout. Of the various units, that on non-metric polyhedrons is almost certainly the most unusual and hence the most experimental.

This unit deals with considerations of polyhedrons not involving the concept of measure. Geometric 0-, 1-, 2-, and 3-dimensional simplexes are defined and, using these concepts, 1-, 2-, and 3-dimensional polyhedrons are studied. The

pupils are asked to build several models of tetrahedrons and cubes. The concept of dimension seems to come out rather clearly in this treatment. A solid cube is observed to be a union of 3-simplexes, its surface to be a union of 2-simplexes. The unit leads into the observation that a simple closed polygon separates a simple surface into two pieces. On models, these two pieces may be observed by a coloring process. Finally the Euler formula is developed in some elementary cases with considerable use being made of models and combinations of models.

It is not yet clear how effectively units of this type can be taught. Many mathematicians will agree, though, with the validity of efforts to improve geometric perception and to emphasize the mathematical importance of reducing complicated configurations to simple elements.

We can well ask ourselves whether or not prospective teachers are now being or are likely to be trained in the understanding and appreciation of concepts like those discussed above. Certainly, more and more set theory will continue to permeate the college mathematics courses for teachers. The concepts, terminology, notation, and use of set theory simplify many aspects of mathematics and give a unity to the language of this subject which has been sorely lacking in the recent past. There seem to be clear-cut trends in the teaching of geometry toward the use of set-theoretic definitions and points of view. We might profitably consider the introduction of a two- or three-week program in a seminar or class for (prospective) teachers which would deal with elementary ideas of simplexes, polyhedrons and dimension.

The Mathematics Speaker's Bureau of Metropolitan New York

The Metropolitan New York Section has formed a bureau which provides a convenient means for high schools to obtain speakers. Professor James N. Eastham of The Cooper Union has enlisted the voluntary services of twenty-seven college and five secondary-school mathematicians to speak on a variety of topics not common to the mathematics classroom. The topics are concerned with basic concepts, structures, fields, and applications of mathematics.

The general aim of this endeavor is to stimulate the interest of high school students in mathematics, to make them realize the need for proper mathematics preparation for careers in teaching and in scientific and technical fields, and to acquaint them with the opportunities open to those with an adequate mathematical training. Booklets listing the speakers and their summarized talks were sent to over 235 high schools. A student group can use the directory in the booklet to contact the desired speaker.

Expenses of the speakers are paid through a National Science Foundation grant made to The Cooper Union which is cooperating with the Metropolitan New York Section of the Association in this project.

Correction

There are presently no plans for the production of the IBM 9902 mentioned in *The national contest in high school mathematics in upper New York state*, by Nura D. Turner, this MONTHLY, vol. 67, 1960, pp. 73–74.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1406. *Proposed by Michael Goldberg, Washington, D. C.*

Cut an obtuse triangle into the least number of acute triangles.

E 1407. *Proposed by Leo Moser, University of Alberta*

Prove that, for $p>3$ and prime, $ab^p - ba^p$ is divisible by $6p$.

E 1408. *Proposed by J. L. Selfridge, IBM, Yorktown Heights, N. Y.*

Find the highest power of 2 which divides the numerator of

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + 1/(2k-1).$$

E 1409. *Proposed by Paul Lukacs, Tel-Aviv, Israel*

In the game of bridge North is dealer. (a) What is the number of the different bidding sequences, assuming that West and East are silent? (b) What is the number of the different bidding sequences if all four players may participate in the bidding?

E 1410. *Proposed by Albert Wilansky, Lehigh University*

Is there a field whose additive and multiplicative groups are isomorphic?

SOLUTIONS

A Hula Hoop Problem

E 1362 [1959, 312]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that, after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the the girl's waist.

Editorial Note. It has been pointed out that Solution I [1959, 918] is faulty. After one revolution of the hula hoop about the girl's waist, the moving point has traveled *half* of a cardioid whose total length is $16a$, where a is the radius of the girl's waist.

Bretschneider's Formula

E 1376 [1959, 588]. *Proposed by V. F. Ivanoff, San Carlos, California*

Show that if A is the area of a quadrilateral having sides a, b, c, d and diagonals e, f then

$$16A^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

I. *Solution by C. F. Pinzka, University of Cincinnati.* Let e_1, e_2, f_1, f_2 be the segments of the diagonals, with e_1, f_1 adjacent to a. Let θ be the angle between e_1, f_1 and let θ' be its supplement. Then

$$16A^2 = 4e^2f^2 \sin^2\theta = 4e^2f^2 - (2ef\cos\theta)^2.$$

But

$$\begin{aligned} 2ef\cos\theta &= 2(e_1 + e_2)(f_1 + f_2)\cos\theta \\ &= 2e_1f_1\cos\theta - 2e_1f_2\cos\theta' - 2e_2f_1\cos\theta' + 2e_2f_2\cos\theta \\ &= -a^2 + b^2 - c^2 + d^2, \end{aligned}$$

by applying the law of cosines to each term. This gives the desired result.

II. *Solution by Joe Lipman, University of Toronto.* Represent the sides and diagonals a, b, c, d, e, f by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ so arranged that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$, $\mathbf{e}=\mathbf{b}+\mathbf{c}$, $\mathbf{f}=\mathbf{a}+\mathbf{b}$. Then

$$16A^2 = 4(\mathbf{e}\times\mathbf{f})\cdot(\mathbf{e}\times\mathbf{f}) = 4(\mathbf{e}\cdot\mathbf{e})(\mathbf{f}\cdot\mathbf{f}) - 4(\mathbf{e}\cdot\mathbf{f})^2 = 4e^2f^2 - 4(\mathbf{e}\cdot\mathbf{f})^2.$$

But

$$\begin{aligned} 2(\mathbf{e}\cdot\mathbf{f}) &= 2(\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}) = -2\mathbf{b}\cdot(\mathbf{c}+\mathbf{d}) + 2\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} \\ &= (\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) - \mathbf{a}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{c} - (\mathbf{b}+\mathbf{d})\cdot(\mathbf{b}+\mathbf{d}) + \mathbf{b}\cdot\mathbf{b} + \mathbf{d}\cdot\mathbf{d} \\ &= a^2 - b^2 + c^2 - d^2. \end{aligned}$$

The result now follows.

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, Leon Bankoff, Shlomo Ben-Adam, D. A. Breault, P. L. Chessin, T. R. Curry, F. J. Duarte, Underwood Dudley, C. E. Franti, George Glauberman, Michael Goldberg, L. D. Goldstone, Larry Grabam, S. H. Greene, G. L. Janusz, Edgar Karst, D. E. Knuth, L. F. Kosinski, T. V. Lakshminarasimhan, D. C. B. Marsh, J. W. Mettler, B. E. Mitchell, E. E. Morrison, F. D. Parker, J. F. Pietenpol, T. A. Porsching, J. M. Rice, Shingu-Tadao, R. Sibson, P. D. Thomas, T. C. Wales, Chung-lie Wang, John Weissman, and the proposer. Late solutions by G. R. Grainger, D. M. J. Hockley, J. M. Kingston, K. N. Kulkarni, and J. P. Phillips.

Editorial Note. The formula of this problem is not new. Pinzka pointed out that it appears as Theorem 8 in an article by G. Dostor entitled "Propriétés nouvelles du quadrilatère en général, etc." in *Archiv der Mathematik und Physik*, vol. 48 (1868), pp. 245–348. It also appears (with proof) in Section 165, pp. 204–205 of Hobson's *Treatise on Plane Trigonometry* (Dover reprint). Another solution appears in "A historically interesting formula for the area of a quadrilateral," by J. L. Coolidge, this MONTHLY, vol. 46, 1939, pp. 345–347.

If the quadrilateral is cyclic, then $ef = ac + bd$ and one easily obtains the Brahmagupta formula

$A^2=(p-a)(p-b)(p-c)(p-d)$, where p is the semiperimeter of the quadrilateral. If the quadrilateral is inscriptible, then $a+c=b+d$ and one obtains the formula $4A^2=e^2f^2-(ac-bd)^2$.

A Simultaneous System of Equations

E 1377 [1959, 588]. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Solve the system of equations

$$u+v=a, \qquad ux+vy=b, \qquad ux^2+vy^2=c, \qquad ux^3+vy^3=d$$

for u, v, x, y.

Solution by R. Sibson, Sutton, Surrey, England. One finds

$$ac-b^2=uv(x-y)^2, \qquad bd-c^2=uvxy(x-y)^2, \qquad ad-bc=uv(x+y)(x-y)^2,$$

whence $(x+y)(ac-b^2)=ad-bc$, $xy(ac-b^2)=bd-c^2$. If $ac=b^2$, then $a/b=b/c=c/d$ and three possible indeterminate solutions arise:

(i) $u=0$, $v=a$, $y=b/a$, x undetermined.

(ii) $v=0$, $u=a$, $x=b/a$, y undetermined.

(iii) $x=y=b/a$, $u+v=a$.

If $ac-b^2\neq 0$, then x and y are the roots of

$$(ac-b^2)t^2-(ad-bc)t+bd-c^2=0,$$

and the values of u and v follow from the first two given equations.

Also solved by A. N. Aheart, Edward Barbeau, Shlomo Ben-Adam, W. J. Blundon, H. S. Bluston, A. P. Boblétt, G. B. Charlesworth, Paul Chessin, G. E. Cooke, F. J. Duarte, Daniel Finkel, George Glauberman, Gerald Goertzel, Michael Goldberg, S. H. Greene, Emil Grosswald, W. O. Hulser, V. F. Ivanoff, G. J. Janusz, M. S. Klamkin, Sidney Kravitz, D. E. Knuth, T. V. Lakshminarasimhan, W. J. Larkin, Joe Lipman, A. E. McGlauchlin, D. C. B. Marsh, C. S. Ogilvy, F. D. Parker, D. J. Persico, C. F. Pinzka, D. C. Stevens, R. G. Wagner, Chung-lie Wang, C. C. Yalavigi, J. W. Young, and the proposer. Late solutions by K. N. Kulkarni and Vencil Skarda.

Klamkin pointed out that this problem, with more variables, appears in *The Collected Papers of S. Ramanujan*, Cambridge, 1927, p. 18. Ramanujan gives a method of solving the system of equations with partial fractions.

The Reunion Problem

E 1378 [1959, 588]. *Proposed by Paul Pargas, Washington, D. C.*

Draw a circle by tracing around a given circular disk D and mark an arbitrary point A on the circle. By drawing circles only with the use of the disk D, where one is permitted to place the rim of D on any two distinct given points, find the point B on the original circle which is diametrically opposite the point A.

Solution by S. H. Greene, RCA, Croydon, Pa. If two equal circles with centers O_1 and O_2 intersect in points P and Q, then $\overrightarrow{O_1P}=\overrightarrow{QO_2}$.

Now let A be any point on circle (O_1) and draw any other circle (O_2) through

A, intersecting (O_1) again at Q. Then $\overrightarrow{O_1A}=\overrightarrow{QO_2}$. Draw any other circle (O_3) through Q intersecting circle (O_2) again at R. Then $\overrightarrow{QO_2}=\overrightarrow{O_3R}$. Draw any other circle (O_4) through R to intersect circle (O_1) at T and circle (O_3) at S. Then $\overrightarrow{O_3R}=\overrightarrow{SO_4}$. Draw the other circle (O_5) through S and T intersecting (O_1) again at B. Then $\overrightarrow{SO_4}=\overrightarrow{O_5T}$ and $\overrightarrow{O_5T}=\overrightarrow{BO_1}$. Thus $\overrightarrow{O_1A}=\overrightarrow{BO_1}$, and B is the point on circle (O_1) diametrically opposite A.

Also solved by D. C. B. Marsh, C. F. Pinzka, R. A. Wiggins, and the proposer. Late solution by L. D. Goldstone.

Editorial Note. Allied to this problem is that of constructing a circle (using the disk D) tangent to a given circle (O_1) (drawn with D) at a given point P. Hans Liebeck remarked that this allied problem was proposed and solved by R. Penrose in the October 1954 issue of *Eureka* (University of Cambridge, England). Problem E 1378 permits us to solve this allied problem. Thus draw any circle (O_2) through P to cut (O_1) again in A. By Problem E 1378, find the point B on (O_2) diametrically opposite A. Now draw the other circle (O_3) passing through P and B. This circle (O_3) is tangent to circle (O_1) at P.

An Abelian Semigroup

E 1379 [1959, 588]. *Proposed by A. A. Mullin, University of Illinois*

Let S be the set of all real numbers and let $a * b$ denote max (a, b), where $a\in S$, $b\in S$. Show that the system $\{S; *\}$ is an abelian semigroup.

Solution by Underwood Dudley, University of Michigan. First, $a * b$ is a real number, so the system is closed under the operation. Second, $(a * b) * c = a * (b * c) = \max(a, b, c)$, so the operation is associative. Third, since $\max(a, b) = \max(b, a)$, $a * b = b * a$, so $\{S; *\}$ is an abelian semigroup.

Also solved by Edward Barbeau, H. F. Bechtell, D. A. Breault, J. L. Brown, Jr., R. F. Brown, Paul Chessin, R. J. Cormier, R. A. Fleck, L. D. Goldstone, Alfred Gray, S. H. Greene, J. C. Hickman, D. E. Knuth, Gerald Liebowitz, Joel Levy and H. J. Lieberman (jointly), Joe Lipman, D. C. B. Marsh, L. D. Olson, F. D. Parker, D. J. Persico, C. F. Pinzka, P. L. Renz, D. R. Rogers, Benjamin Sims, D. C. Stevens, Chung-lie Wang, C. O. Wilde, and the proposer. Late solutions by F. M. Ali and N. A. Halim (jointly), George Glauberman, Bro. Joseph Heisler, and R. T. Hood.

Editorial Note. If $a\,|\,b$ denotes min (a, b), then $\{S; |\}$ is also an abelian semigroup, and the system $\{S; *, |\}$ obeys both distributive laws. That is, $\{S; *, |\}$ is (see, *e.g.*, H. Levi, *Elements of Algebra*) a "number system," in which either operation can serve as "addition" and the other as "multiplication." If $T\subset S$, then $\{T; *, |\}$ is also a number system, and if T has a smallest element z and a largest element e, then z and e are the identity elements for $*$ and $|$ respectively.

A Matrix Problem

E 1380 [1959, 588]. *Proposed by D. S. Passman, Polytechnic Institute of Brooklyn*

Let A and B be $n\times n$ nonsingular matrices with $A+B=kE$, where k is a scalar and E is a matrix all of whose elements are 1's. If $S(C)$ denotes the sum of the elements of C, show that

$$\{1 - kS(A^{-1})\}\{1 - kS(B^{-1})\} = 1.$$

Solution by Joe Lipman, University of Toronto. Since $ECE=\{S(C)\}E$, we

have $\{1-kS(A^{-1})\}E=\{I-kEA^{-1}\}E=\{I-(A+B)A^{-1}\}E=-BA^{-1}E$. Therefore

$$\{1-kS(A^{-1})\}\{1-kS(B^{-1})\}E=(-BA^{-1})\{1-kS(B^{-1})\}E$$
$$=(-BA^{-1})(-AB^{-1})E=E.$$

The conclusion is immediate.

Also solved by Frank Ayres, Jr., Edward Barbeau, R. F. Brown, Lin-chuan Cha, B. Deaville and J. Hooley (jointly), Seymour Geisser, S. H. Greene, John Jordan, P. G. Kirmser, Alphia E. Knapp, D. E. Knuth, P. S. Landweber, D. C. B. Marsh, F. D. Parker, D. J. Persico, C. F. Pinzka, J. R. Swenson, Chung-lie Wang, J. V. Whittaker, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4893. *Proposed by Joe Lipman, University of Toronto*

Let A, B be positive integers, A odd, B even, and let $A^2+B^2=p$, a prime.

(a) Show that, except perhaps for sign, A and B are respectively the real and the imaginary parts of S, where

$$S=\sum_{x=0}^{\frac{1}{2}(p-1)}\exp\left\{\tfrac{1}{2}\pi i \operatorname{ind}(x^4+a^4)\right\},$$

with a any nonzero, fixed quadratic residue (mod p).

(b) Let R be any fixed quadratic residue (mod p) and N any fixed nonresidue. Prove

$$A=\sum_{x=0}^{\frac{1}{2}(p-1)}\left(\frac{x^4+R}{p}\right),\qquad B=\sum_{x=1}^{\frac{1}{2}(p-1)}\left(\frac{x^4+N}{p}\right),$$

where the summands are Legendre symbols.

4894. *Proposed by Olga Taussky, California Institute of Technology*

Let A, B be two positive definite hermitian matrices which can be transformed simultaneously by unitary transformation to diagonal forms of similarly ordered numbers. Let x be any vector of complex numbers. Show that $(Ax, x)(Bx, x)\leqq(ABx, x)(x, x)$, and discuss the case of equality. Two sets of n

positive numbers, $\{a_i\}$, $\{b_i\}$ are called similarly ordered if $(a_i-a_k)(b_i-b_k)\geqq 0$ for all $i, k=1, \cdots, n$.

4895. *Proposed by E. S. Keeping, University of Alberta*

γ being Euler's constant, prove that

$$\int_0^\infty (\log x)^2 e^{-x}dx = \tfrac{1}{6}\pi^2 + \gamma^2.$$

4896. *Proposed by H. S. Shapiro, New York University*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$, where the a_n lie in the interior of a Jordan curve Γ in the complex plane, the origin being exterior to Γ. Prove that there exist singularities $z_1, z_2, \cdots, z_k$ $(k\geqq 1)$ on the unit circle, and positive integers $n_1, \cdots, n_k$ such that $z_1^{n_1}\cdots z_k^{n_k}=1$.

4897. *Proposed by D. J. Newman, Brown University*

Let $F(x, y)$ be such that: for any fixed y, $F(x, y)$ is a polynomial in x, and for any fixed x, $F(x, y)$ is a polynomial in y. Prove $F(x, y)$ is a polynomial in x and y.

4898. *Proposed by R. C. Reed, University College of the West Indies*

A sequence of n digits, each digit being a 1, 2 or 3, will be called an A_n-sequence. The six permutations of 123 are the only proper A_3-sequences, and for $n>3$ a proper A_n-sequence is defined recursively as follows: an A_n-sequence $(n>3)$ is proper if, and only if, it can be obtained from some proper A_{n-1}-sequence by deleting a digit of the A_{n-1}-sequence and putting in its place the other two digits (in either order). Thus the sequences 213, 2233, 31233 are examples of proper A_n-sequences.

Show that the number of distinct proper A_n-sequences is $\frac{3}{4}[3^{n-1}-2-(-1)^n]$.

SOLUTIONS

Inverse Functions

4827 [1959; 67, 820]. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y = a_1x + a_3x^3 + a_5x^5 + \cdots,$$

find a function y knowing that if we invert the series we get

$$x = a_1y - a_3y^3 + a_5y^5 - \cdots.$$

III. *Note by Julius Lieblein, David Taylor Model Basin, Washington, D. C.* Previous solutions involve special cases with doubt expressed as to the possibility of a general form.

Starting with the notation and analysis of I, we write (2) p. 821 (November 1959) as $tF(t) = -(1/t)F(1/t)\equiv H(t)$, say, where $t=y/x$. Thus, $H(t)$ is any func-

tion satisfying $H(t)+H(1/t)\equiv 0$. The substitution $t=\exp u$ shows that H is any odd function of u, $u=\log t=\log(y/x)$, say $g(u)$. To avoid singularities, we may limit consideration to odd exponential or hyperbolic functions of $\log t$. Recalling that $F(t)=x^2$, we have $H(t)=tF(t)=xy$. A general form of solution for y as a function of x is therefore given (implicitly) by

$$(*) \qquad xy = g(\log(y/x)) = g(u),$$

where the odd function g is any function subject to the above limitation. A more direct form is $y=xp(x^2)$ where the function p is given by

$$(**) \qquad p^{-1}(t) = (1/t)g(\log t).$$

To see this, note that we have from (*) $g(\log t)=xy=tx^2=tp^{-1}(t)$. Hence $t=y/x=p(x^2)$. This includes the two solutions given previously and leads readily to others.

Solution I, by Kokar, is obtained from

$$g(u) = 2^{2n+1}k^2(\sinh u)^{2n+1}, \qquad y = x(1 - x^2/k^2)^{-1/2} \text{for } n = 0.$$

The latter is, incidentally, the case with the simplest odd hyperbolic function of $\log t$. Solution II, by Carlitz, namely, $y=\operatorname{gd} x$, may be obtained from (**) by defining the function p by $p(z^2)=(\operatorname{gd} z)/z$.

Other interesting special solutions are:

$$\text{(a)} \quad g(u) = \tanh \tfrac{1}{2}u, \qquad xy = (y-x)/(y+x), \qquad p^{-1}(t) = (t-1)/t(t+1).$$

The series for x or y in terms of the other is obtained from the root of the appropriate quadratic equation.

$$\text{(b)} \qquad g(u) = \log\tanh(\tfrac{1}{2}\log u) = \log\left(\frac{\log t - 1}{\log t + 1}\right).$$

It is apparent that explicit solutions for y or x in terms of the other will be very rare.

Definite Integral Evaluation

4830 [1959, 147]. *Proposed by D. B. Larson, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Evaluate the integral

$$\int_{-1}^{1} e^{iAx}(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx,$$

where A and B are positive, real constants, $B<1$ and $i^2=-1$.

Solution by E. J. Scott, University of Illinois. Let $I(A, B)$ be the given integral. In view of the Euler relation $e^{iAx}=\cos Ax+i \sin Ax$ and the fact that $\sin Ax$ and $\cos Ax$ are odd and even functions, respectively, we have the result

$$(1) \qquad I(A, B) = 2\int_0^1 \cos Ax(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx.$$

Now,

$$(2) \qquad \cos Ax/(1 - Bx^2) = 1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k B^{n-k}A^{2k}/(2k)!\right]x^{2n},$$

which is convergent in the interval of integration. Therefore,

$$(3) \quad I(A, B) = \pi + 2\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k B^{n-k}A^{2k}/(2k)!\right]\int_0^1 x^{2n}(1 - x^2)^{-1/2}dx.$$

But,

$$(4) \quad \int_0^1 x^{2n}(1 - x^2)^{-1/2}dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot\cdots\cdot(2n-3)(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n-2)(2n)}, \quad n = 1, 2, \cdots.$$

Consequently, from (3) we obtain

$$(5) \quad I(A, B) = \pi\left\{1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\frac{1\cdot 3\cdot\cdots\cdot(2n-3)(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n-2)(2n)}\cdot\frac{B^{n-k}A^{2k}}{(2k)!}\right]\right\}$$

The result (5) holds for nonnegative A and $0 \leqq B < 1$.

Also solved by J. W. Brown, L. Carlitz, Y. L. Luke, O. E. Stanaitis, W. F. Trench, and the proposer and J. L. Lotsoff.

Editorial Note. The final form differed from solution to solution. Most of the solvers preferred to avoid the double series by introducing Bessel's functions or hypergeometric functions. Perhaps the most compact is the proposer's $I(A, B) = \pi\sum_{n=0}^{\infty}(-B)^n(d^{2n}J_0(A)/dA^{2n})$.

Poisson Distribution

4841 [1959, 317]. *Proposed by John Lamperti, California Institute of Technology*

Let N be a random variable whose values are nonnegative integers. Independently of each other, each of N balls is placed either in urn A with probability p $(0<p<1)$ or in urn B with probability $1-p$, resulting in N_A balls in urn A and $N_B = N - N_A$ in urn B. Show that the random variables N_A and N_B are independent if and only if N has a Poisson distribution.

Solution by William Kruskal, University of Chicago. By hypothesis,

$$\Pr\left[N_A = a \quad \text{and} \quad N_B = b \,\middle|\, N_A + N_B = n\right] = \binom{n}{a}p^a q^b$$

for $a+b=n$, a, b, n nonnegative integers, and $q = 1-p$. Hence

$$\Pr\left[N_A = a \quad \text{and} \quad N_B = b\right] = \frac{(a+b)!}{a!b!}p^a q^b \pi_{a+b},$$

where $\pi_n = \Pr[N=n]$.

If N has a Poisson distribution, *i.e.* if, for some λ, $\pi_n = e^{-\lambda}\lambda^n/n!$, it is immediate that $\Pr[N_A=a \text{ and } N_B=b]$ factors and that N_A and N_B are independent.

If N_A and N_B are independent, then $\pi_{a+b}(a+b)! = f(a)g(b)$ for some functions f and g. Note that neither $f(0)$ nor $g(0)$ can be zero, for there is positive probability that $N_A=0$ and that $N_B=0$. In short there is a function h such that $f(a)g(b) = h(a+b)$ for a, b nonnegative integers. Hence

$$f(1)g(0) = f(0)g(1), \qquad f(2)g(0) = f(1)g(1) = f(0)g(2), \text{ etc.}$$

and, by easy induction,

$$f(a) = f(1)[g(1)/g(0)]^{a-1}, \qquad g(b) = g(1)[f(1)/f(0)]^{b-1}.$$

But the quantities in square brackets are equal. Hence we may write, for some α, α', λ,

$$f(a) = \alpha e^{\lambda a}, \qquad g(b) = \alpha' e^{\lambda b}, \qquad \pi_{a+b} = \alpha\alpha' \frac{e^{\lambda(a+b)}}{(a+b)!}$$

so that N has a Poisson distribution.

Also solved by D. R. Brillinger, Luciano Daboni, N. J. Fine, George Glauberman, Emil Grosswald, D. A. Jones and J. C. Hickman, C. E. Lemke, D. C. B. Marsh, R. S. Pinkham, John Rainwater, G. S. Rogers, W. F. Trench, and the proposer.

Density of Random Variable

4842 [1959, 317]. *Proposed by John Lamperti, California Institute of Technology*

Let X be a nonnegative random variable, and let Y be uniformly distributed on the interval $(0, X)$. Let $Z = X - Y$. Show that Y and Z are independent if and only if X has the density

$$f(x) = \begin{cases} a^2 x e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \qquad \text{for } a \geq 0.$$

Solution by N. J. Fine, Institute for Advanced Study. Let $F(x) = \Pr[X \leq x]$, $G(y) = \Pr[Y > y]$. Then

$$G(y) = \int_{y^+}^{\infty} \left(1 - \frac{y}{x}\right) dF(x) \qquad (y \geq 0).$$

Clearly $\Pr[Z > z] = G(z)$ for $z \geq 0$. Also, for y, $z \geq 0$,

$$\Pr[Y > y, Z > z] = \int_{(y+z)^+}^{\infty} \left(1 - \frac{y+z}{x}\right) dF(x),$$

since $Y > y$, $Z > z$ are equivalent to $X = Y + Z > y + z$ and $y < Y < X - z$. Thus the independence of Y and Z is equivalent to $G(y+z) = G(y)G(z)$. It follows that

$G(y)=G(1)^y$ for rationals $y>0$. Since G is monotone, this holds for all $y>0$. If $G(1)=0$, then $G(y)=0$ for all $y>0$, which implies that $\Pr[Y=0]=\Pr[Z=0]=\Pr[X=0]=1$.

Leaving aside this trivial case, we may write $G(y)=e^{-ay}$ for some $a>0$. Now, for $h>0$,

$$G(y+h)-G(y)=-\int_{y^+}^{y+h}\left(1-\frac{y}{x}\right)dF(x)-h\int_{(y+h)^+}^{\infty}\frac{1}{x}\,dF(x).$$

The first integral does not exceed, in absolute value,

$$(h/y)\int_{y^+}^{y+h}dF(x)=(h/y)\Pr[y<X\leqq y+h]=(h/y)o(1).$$

Thus,

$$-ae^{-ay}=\lim_{h\to 0^+}\frac{G(y+h)-G(y)}{h}=-\int_{y^+}^{\infty}\frac{1}{x}\,dF(x)$$

$$=\frac{1}{y}\left(e^{-ay}-\int_{y^+}^{\infty}dF(x)\right)=\frac{1}{y}\left(e^{-ay}-1+F(y)\right).$$

Solving for $F(y)$, we get $F(y)=1-(1+ay)e^{-ay}$, $F'(y)=a^2ye^{-ay}$ $(y\geqq 0)$.

The converse is easily established.

Also solved by D. R. Brillinger, George Glauberman, Emil Grosswald, D. A. Jones and J. C. Hickman, W. S. Lawton, M. D. Mavinkurve, Roger Pinkham, John Rainwater, G. S. Rogers, W. F. Trench, and the proposer.

Generators of the Additive Group of a Field

4847 [1959, 427]. *Proposed by P. T. Bateman, University of Illinois*

Suppose f is a nonconstant polynomial function over a field K. Show that the additive group of K is generated by the values taken on by f provided either (a) K is of characteristic zero or (b) K has prime characteristic p and f has degree less than p. Also show by an example that some restriction such as (a) or (b) is needed.

Solution by N. J. Fine, The Institute for Advanced Study. Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots$, $a_n\neq 0$. Then the additive group G generated by the values of f contains $\Delta^{n-1}f(x)-\Delta^{n-1}f(0)=n!a_nx$ $(x\in K)$. Either assumption, (a) or (b), guarantees that $n!a_n\neq 0$, so we can replace x by $(n!a_n)^{-1}y$, to get $y\in G$ for all $y\in K$.

The example required is $K=Gf(4)$, with $f(x)=x^2+x+1$. The values assumed by f are 0 and 1, so $G=\{0,1\}\neq K$.

Also solved by the proposer.

A Ratio of Integrals

4848 [1959, 427]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Massachusetts*

Without performing any integration determine the ratio

$$\int_0^1 \frac{dt}{\sqrt{(1-t^4)}} : \int_0^1 \frac{dt}{\sqrt{(1+t^4)}}.$$

Solution by A. B. Farnell, Convair Research Laboratory, San Diego, California. Since the first integral involved is convergent, and

$$\int_0^1 \frac{dt}{\sqrt{(1-t^4)}} = \int_0^{\frac{1}{2}\pi} \frac{d(\sqrt{(\sin 2\theta)})}{\cos 2\theta} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(\sin 2\theta)}},$$

$$\int_0^1 \frac{dt}{\sqrt{(1+t^4)}} = \int_0^{\frac{1}{2}\pi} \frac{d(\sqrt{(\tan\theta)})}{\sec\theta} = \frac{1}{\sqrt{2}} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(\sin 2\theta)}},$$

the desired ratio is $\sqrt{2}$.

Also solved by C. H. Cunkle, R. D. Depew, P. H. Diananda, P. G. Engstrom, H. E. Fettis, N. J. Fine and M. Jerison, Donald Garlock, S. Glusman, H. W. Gould, Cornelius Groenewoud, Emil Grosswald, J. M. Harding, L. S. Kennison, Brockway McMillan, David Moskowitz, C. S. Ogilvy, C. F. Pinzka, G. E. Raynor, E. T. Sheffield, W. C. Teachout, Jr., F. R. Urbanus, Chih-yi Wang, Alan Wayne, C. C. Yalavigi, David Zeitlin, and the proposer.

Editorial Note. Several solvers used contour integration and transformations of the complex plane. In this way Raynor and the proposer obtained the more general result:

$$\int_0^1 dt/(1-t^{2n})^{1/n} : \int_0^1 dt/(1+t^{2n})^{1/n} = \sec(\pi/2n), n = 2, 3, \cdots.$$

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officeers of the Association.

Plane Trigonometry (2nd ed.). By Gordon Fuller. McGraw-Hill, New York, 1959. xiii+281 pp. $4.75.

This edition differs from the original (reviewed in this MONTHLY, vol. 58, 1951, pp. 645–647) primarily in that more emphasis is placed on analytical trigonometry, more exercises are supplied, the order of topics is slightly rearranged (and the price has been increased by $2).

MELVIN HENRIKSEN
Purdue University

College Algebra (4th ed.). By J. B. Rosenbach, E. A. Whitman, B. E. Meserve, P. M. Whitman. Ginn, Boston, 1958. xiv+564 pp. $5.50.

Essentials of College Algebra (2nd ed.). By J. B. Rosenbach, E. A. Whitman, B. E. Meserve, P. M. Whitman. Ginn, Boston, 1958. xii+368 pp. $4.75.

Teachers will be familiar with earlier editions of these texts. Limited changes in subject matter have been made. Elementary notions of statistics, the use of matrices in solving linear systems and summation notation are among the new topics introduced. The shorter of the texts is an abbreviated version of the other. The abbreviation has been effected primarily by omission of certain topics and exercises.

These texts are conventional. On page one the authors remark that the printed symbols 1, 2, 3, · · · are *numerals* rather than *numbers*. However, no serious attempt is made to distinguish between mathematical symbols and the concepts represented by these symbols.

Some of the field postulates for addition and multiplication are stated, *but are not used deliberately to develop computational techniques*. Rather the student is told how to manipulate symbols so as to arrive at approved answers. A remark concerning the fundamental assumptions seems unfortunate. The student is told that these assumptions assert that "*basic properties of numbers in arithmetic hold also for literal number symbols.*"

In the section on functions the authors define a function to be the dependent variable. Since a variable has been defined to be a symbol this leads to logical difficulties.

The format is attractive and exercise lists are adequate. Many teachers who wish to teach traditional college algebra using traditional terminology will find these texts satisfactory.

CHARLES BRUMFIEL
Ball State Teachers College

Intermediate Algebra. By Ross H. Bardell and Abraham Spitzbart. Addison-Wesley, Reading, Massachusetts, 1959. ix+274 pages. $4.75.

This book is written in the traditional style for "the student who begins his college mathematics work with inadequate preparation in mathematics." Such students seem to need (and to profit from) long lists of routine drill problems —this text has long lists of such problems. Many of these inadequately prepared students do not seem to be aware that sometimes an algebraic identity can be used either to multiply or to factor, hence the authors list such identities twice, *e.g.*, $(u+v)(u^2-uv+v^2)=u^3+v^3$ in a list of multiplication formulas and $u^3+v^3=(u+v)(u^2-uv+v^2)$ in a list of factors.

The print on one side of a page shows through on the other side in an annoying fashion.

ROBERT E. GREENWOOD
University of Texas

Analysis and Control of Nonlinear Systems. By Y. H. Ku. Ronald Press, New York, 1958. vii+360 pp. $10.00.

Problems giving rise to nonlinear differential equations derived from consideration of mechanical vibrations and electrical oscillations are here studied in detail using both analytical and topological approaches. By use of the topological approach general system behavior in the neighborhood of singular points is determined, thus largely bypassing the need for numerical solutions of the differential equations. Only brief mention is made of the possibilities for solution of nonlinear systems on analog and digital computers. The last two chapters are devoted to the analysis and control of nonlinear feedback systems. The book was written as a graduate text in nonlinear circuit analysis, and the inclusion of some 120 problems for solution helps in fulfilling this aim.

The bibliography of 735 items in chronological order beginning with the year 1860, spanning periods of 40 years, 20 years, 10 years and yearly from then on seems most adequate. An alphabetical list of authors is lacking.

The author presents striking evidence that nature is nonlinear, that nonlinearities may not only be useful but necessary, and that failure to recognize the reality of nonlinearities may lead to catastrophes such as the tragic collapse of the Tacoma suspension bridge.

The book is well written and illustrated, and is a particularly good book for the mathematician interested in nonlinear problems.

HERBERT A. MEYER
University of Florida

Applied Mathematics for Engineers and Scientists. By C. G. Lambe. Macmillan, New York, 1959. xii+518 pp. $8.50.

This book was apparently written for a specific course of study in British universities. It is hard to see how it can be fitted very snugly to current traditions in American curricula. Largely devoted to the subject of mechanics, its introductions to the various concepts and methods of mechanics will be found too elementary for physicists, too brief for engineers, and too numerous for differential equaters. The principal merit of the book lies in its many thoroughly worked examples, which students and instructors in any of the above mentioned courses would find handy to have on their bookshelves. The authors have, in this respect, done an excellent job of editing their notebooks. There is, of course, a great deal to be said in favor of a long parade of examples as a teaching device. One also suspects that there was difficulty in choosing a title, with an ultimate surrender to the current fad of forming a random combination from the words "applied," "engineer," "methods," "advanced," "mathematics" and "scientist." More informative would have been "Solved: One Thousand Problems in the Mechanics of (Rigid or Deformable) Bodies."

FRANCIS SCHEID
Boston University

Méthodes Numériques. By J. Kuntzmann. Dunod, Paris, 1958. 252 pp. 3,600 F. (About $8.60).

This book provides thorough coverage of the first chapters of numerical analysis: difference calculus, numerical interpolation and differentiation. Aimed toward the practical computer, there are many fine tabular layouts, summaries, computational schemes and tips, graphs, and worked examples. Though the book is, on the whole, elementary, and the bulk of the theory standard, the author is alert to the literature of the past 15–20 years and in addition to many recent references has included a number of new things which are of considerable utility. His treatment of errors via kernels, and interpolation with nonlinear families are particularly noteworthy.

P. J. Davis
National Bureau of Standards, Washington, D. C.

Mathematics of Physics and Modern Engineering. By I. S. Sokolnikoff and R. M. Redheffer. McGraw-Hill, New York, 1958. viii+810 pp. $9.50.

This voluminous work is addressed primarily to engineering students and practicing engineers. Both of these groups face a dilemma which confronts most scientific workers, students and practitioners alike—the subjects and topics which they need to know thoroughly are too numerous and extensive to be learned in the time available. An inescapable implication of this predicament is that either depth or extent will have to be sacrificed to a considerable degree. Whether the best solution is to study a few basic subjects intensively or to work entirely toward breadth of scope will depend upon the individual's capacity for the abstract and precise thinking necessary for intensive study. In either case there is a need for easy reading surveys which present, without pretense of precision and completeness in all details, the rather formidable miscellany of topics fundamental and applied which make up the complex of modern industrial mathematics. Although there are facets of this book which run counter to the reviewer's prejudices, it is the reviewer's opinion that this work does present for the engineer who has finished the elementary calculus a comparatively easy and yet satisfactory introduction to a variety of topics which have been judiciously selected.

For the majority of engineers at the junior-senior level, the expositions will be found to tie in closely with their previous mathematical training both with regard to the points of departure and the difficulty of the developments. For the most part, these gradual transitions are definitely in order since the book is designed for readers who must budget their time strictly. However, in one or two instances the reviewer would have preferred more modern formulations than those presented despite the resulting break in the continuity with the typical prerequisite engineering mathematics. Specifically, complex numbers are presented as numbers represented by $a+ib$; the concept infinite series is introduced by the statements: "A series is a sum of terms" $\cdots$ "$a_1+a_2+\cdots+a_n$

is a series consisting of n terms. An infinite series is a series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1.1}$$

which has infinitely many terms." The reviewer is prejudiced in favor of the explicit definition of the concept of complex number as an ordered pair of real numbers subject to the well-known definitions of equality, sum, and product and to the definition of an infinite series as a pair of infinite sequences—a primary sequence $\{a_n\}$ and a derived sequence $\{s_n\}$, $s_n = a_1 + \cdots + a_n$. In addition, there is an occasional recourse to the indefinite "small"—but these matters should be considered with the purpose of the book in mind.

The chapter headings and some of the numerous topics treated are: (1) Ordinary Differential Equations (the usual elementary topics together with an excellent set of applications); (2) Infinite Series (tests for convergence, uniform convergence, Taylor's series, complex series, Fourier series); (3) Functions of Several Variables (partial differentiation, the chain rule, directional derivative, maxima and minima, Taylor's formula, calculus of variations, surface integrals); (4) Algebra and Geometry of Vectors, Matrices (vector algebra including the dot and cross products, differentiation, applications to analytic geometry and mechanics, linear vector spaces, transformations of base vectors); (5) Vector Field Theory (curvilinear coordinates, the metric quantities g_{ab}, the Gauss divergence theorem, Stokes's theorem, applications to heat flow and fluid mechanics); (6) Partial Differential Equations (the vibrating string, solution by series, solution by integrals, elliptic, parabolic, and hyperbolic equations); (7) Complex Variable (analytic functions, integration, Cauchy's integral theorem and integral formula, Taylor's series, Laurent's expansion, residues, conformal mapping, applications); (8) Probability (elementary topics, random variables and expectation, discrete and continuous distributions, probability and relative frequency, the Poisson law, theory of errors, variance); (9) Numerical Analysis (graphical methods, Newton's method, iterative methods, interpolation, empirical formulas, least squares, harmonic analysis, numerical integration of infinite series).

Altogether there are some 214 sections. Considering its size the book is remarkably free of misprints.

HOMER V. CRAIG
Boeing Airplane Company and The University of Texas

Ancillary Mathematics. By H. S. W. Massey and H. Kestelman. Pitman, London, 1959. xxvi+990 pp. 75 s. (about $10.54).

This rather large book is aimed at giving underclassmen college students in physics and chemistry a thorough preparation for the type of mathematics they will need in order to develop an understanding of the physical sciences. In this direction, the authors appear to have especially in mind the requirements of thermodynamics, statistical mechanics, and quantum theory. The book contains thirty-two chapters—starting with the concept of a limit and basic properties of

the real numbers and eventually covering, among other things, most of the topics traditionally given (at American universities) in a four year course on engineering mathematics. The five chapters on mechanics (18, 19, 20, 24, and 25) together with the applications in Chapter 26 (introduction to Fourier series) and Chapter 32 (introduction to the variational calculus) forms the basis of an excellent undergraduate course in mechanics. Since the authors wrote the book with care and precision, it does not have the usual mathematical faults that are typical in books of this nature. The book contains a large number of illustrative examples and exercises.

PASQUALE PORCELLI
Mathematics Research Center
University of Wisconsin

Tauberian Theorems. By H. R. Pitt. Oxford, New York, 1959. 174 pp. $6.00.

Since the classical theorem of Tauber, there has been a continuous and widespread investigation of Tauberian theorems and Tauberian conditions. In the preface the author defines the scope of this book as including the topics and methods which follow most naturally from the work of Hardy, Littlewood and Wiener. These topics and methods have formed a most fruitful part of the theory. Chapter headings are: Introduction, Elementary Tauberian Theorems, Classical Tauberian Theorems, Wiener's Theory, Mercerian Theorems, Tauberian Theorems and the Prime Number Theorem. The chapter on the Prime Number theorem compares the Landeau-Ikehara method, the classical proofs, and the elementary proofs of Selberg and of Erdös. The chapter on Mercerian theorems presents a unified treatment of that subject.

This is clearly a monograph that should be in the hands of anyone interested in the subject of summability, especially those wishing to do research on Tauberian theorems. At the beginning of each chapter is a brief section discussing the results that are to follow. The subject matter of the chapter is illuminated against its proper background and the reader is shown where the material fits in with the rest of the theory. The feasibility of other possible developments is examined so that very soon the reader has an appreciation of where the problems lie and the hazards that will be encountered in attempting their solution. Similar brief remarks appear when appropriate at the beginning of some sections. The reviewer found these brief discussions one of the most instructive and thought provoking features of the book. Nor was the beauty of the formal development diminished, but rather enhanced, by these touches of the author's insight.

This is the second monograph published by the Tata Institute of Fundamental Research, Bombay and it should be congratulated on bringing such a valuable work before the mathematical public.

GORDON M. PETERSEN
University College, Swansea

Mathematics is Easy. By D. S. Watt. Princes Press, London, 48/- and St. Martin's Press, New York. 488 pp. $10.00.

In the foreword, E. H. Lloyd describes the author's purpose as follows: "He sets out to produce a book from which any literate person could teach himself mathematics; that is to say, not just generalities, but real detailed technical mathematics." The author writes that the book "aims at a solid foundation of mathematical knowledge" and belies his title by maintaining: "Hard work and hard thinking are essential to any kind of mathematical knowledge."

The topics covered in the book are rudiments of plane and solid geometry—mostly intuitive—the usual content of a college algebra course, with a smattering of calculus; there is a little on matrices and trigonometry. The style is easy-going and should be intelligible to a determined layman. The foundation laid in elementary algebra is for the most part adequate, but that in calculus is shaky to say the least. For instance, after defining the indefinite integral to be the anti-derivative, he defines the definite integral as follows: "Where the integral has limits or extreme values of the variable, we are said to have a definite integral."

The most serious defect in the book is that it contains no exercises. Though the reader is invited to read carefully, he is not given the opportunity of discovering something himself. But in spite of the title, there is much in this book that is worth working at.

BURTON W. JONES
University of Colorado

Numerical Analysis and Partial Differential Equations. Volume V, *Surveys in Applied Mathematics.* By George E. Forsythe and Paul C. Rosenbloom. Wiley, New York, 1958. x+204 pp. $7.50.

This book consists of two papers, *Contemporary State of Numerical Analysis* by George E. Forsythe, and *Linear Partial Differential Equations* by Paul C. Rosenbloom. One fifth of this book is devoted to a brief survey of numerical analysis with major emphasis on methods devloped for automatic digital computation. The remainder is concerned primarily with advances in the theory of linear partial differential equations in the last ten years and gives many applications to problems in applied mathematics. In both of these reports the status of the work done and to be done is clearly stated and special attention has been given to Russian publications.

LAWRENCE A. WELLER
Monsanto Chemical Company, Miamisburg, Ohio

Intermediate Algebra. By L. M. Kells. Prentice-Hall, Englewood Cliffs, New Jersey, 1959. xiv+296 pp. $5.25.

The author states that "This text is designed primarily for college students who have inadequate training in algebra." Content is so chosen and ordered that the text is adaptable to fairly diverse usage in colleges.

The most distinguishing feature of the book is a precision of statement all too often lacking in such texts. This is particularly welcome on fractional exponents and radicals. The treatment of the "Laws" of algebra is unusually well done. The balance between extreme rigor and intuition is very suitable for this level text. Appropriately for an intermediate algebra text, the elementary processes and signed numbers are treated reasonably briefly but at a more sophisticated level than is customary in elementary texts. Particularly well done is the explanation of operations with signed numbers.

"Topics for discussion" are not necessarily supplementary but represent a "special treatment" that should prove pedagogically useful. Although these topics are generally briefly presented, they are sufficiently clear that many students should be able to follow them easily.

Actual flaws in the text are few. This reviewer does question the desirability of two forms of the quadratic formula (one for roots involving i). The latter form cannot be derived from the author's previous material, although it can be verified by substitution. A slight, but highly desirable, modification in previous definitions would avoid this. This reviewer would have liked to have seen the Factor and Remainder theorems included as well as some problems on reducible trinomials.

The careful exposition and numerous well chosen problems and exercises should serve to provide the student with ample opportunity to practice manipulative skill while learning something of the logical nature of algebra. Illustrative examples are carefully worked out and well explained. This text should make algebra appear interesting and useful in its own right while serving as a stimulating introduction to more mathematics.

JAMES L. SIMPSON
Montana State College

Foundations of Mathematics. By Carl H. Denbow and Victor Goedicke. Harper, New York, 1959. xviii+613 pp. $6.00.

Here is an exciting book with a new approach in pedgagogy. One of the chief objectives is a gradual transition from an empirical approach to a deductive treatment. The authors build the reader's confidence and interest in early sections, and then point up limitations of the experimental approach to pave the way for a logical foundation. Parts I, II, III are entitled respectively Empirical Foundations, Logical Foundations, and Developments and Applications.

As an introduction to Part I, there is an illuminating survey of basic ideas of arithmetic and algebra, including properties of clock numbers, the meaning of associative and distributive laws, and some concepts concerning heat flow in one and two dimensions. Part I begins with modular numbers and proceeds to cover the major part of what is usually styled intermediate algebra. The authors pique the curiosity of the reader by suggesting extensions of given situations. Also, they include some philosophy of mathematics with a view toward inducing

a more flexible and open-minded attitude on the part of the reader.

As an instance of a branch of mathematics in which symbols do not represent numbers, Part II begins with the furniture mover's story which allows introduction of many concepts of group theory. The chapter on rational numbers and fields digs deeper into algebra and throws more light on notions introduced previously. Adequate coverage of logarithms, progressions, and quadratic equations is found in the chapter on real and complex numbers. The final chapter of Part II gives a clear account of logic algebra and set algebra.

In Part III, following a chapter on probability and a brief chapter on statistics, there is a helpful chapter on problem solving with hints on how to transform verbal statements to symbols. There follow 56 pages on trigonometry, a brief but adequate treatment. Some of the usual topics of analytic geometry are treated in 55 pages, and certain topics from the calculus are covered in 83 pages. Here, as elsewhere, the authors point out the need for more powerful mathematics, by means of an example from the experience of the reader. A novel feature here is the introduction of tabular derivatives. The final three chapters consider number theory, "infinity and beyond," and "what is mathematics?" The bibliography for further reading, the tables, and the answers to the odd-numbered exercises add to the usefulness of the book.

This carefully written comprehensive volume is adaptable for use in a wide variety of courses. It should be read by all teachers of mathematics in high school and in the first year of college.

C. E. SPRINGER
University of Oklahoma

Coordinate Geometry with Vectors and Tensors. By E. A. Maxwell. Oxford University Press, London, 1958. xii+198 pp. $4.00.

The author aims to present the minimum amount of coordinate geometry of three dimensions, which he considers necessary for the young mathematician, together with some vectors and tensors. Believing that coordinate methods are essential and that a background should be laid for the application and appreciation of vectors, he presents planes and straight lines first. This preparation permits a presentation of vectors with an algebraic basis. The chapter on tensors is largely confined to the very useful δ_{ij} and ϵ_{ijk}. There follow chapters on spheres, central quadrics, paraboloids, and the general quadric. Such topics as Joachimstal's ratio equation, coaxial spheres, tangents, poles and polars, diameters, centers, and generators are treated. Two treatments of the general quadric are given, one ordinary and the other with the double suffix notation of tensors.

The work throughout is carefully organized. The level is above the elementary. The presentation is clear and rigorous. The exercise lists abound with excellent problems. The author accomplishes his stated aims.

EARL LAFON
University of Oklahoma

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, New York, 1959. 95 pp. $2.95.

By definition, "the fallacy leads by guile to a wrong, but plausible conclusion; the howler leads innocently to a correct result."—This small book is devoted essentially to fallacies. At the end, a few howlers, apparently encountered in class, are included. We mention only one: to find largest angle of triangle with sides 4, 7, 9. Solution: $\sin C = 9/7 = 1.2857$, $\sin 90° = 1$, $\sin 16°36' = .2857$; hence $\sin (90° + 16°36') = 1.0000 + .2857$, $C = 106°36'$ (correct answer). In the discussion the class of triangles is characterized for which this method of solution holds.

The fallacies are taken from the fields of algebra, trigonometry, analytic geometry and calculus. The theory of sets is on the whole disregarded, thus eliminating a fruitful source of good fallacies.

The level of the fallacies, as well as the level of the explanations, varies within wide limits. The fallacy that every triangle is isosceles is taken up in three different forms. The analysis of one of these occupies nine pages and leads to Pasch's axiom, Ptolemy's theorem on four concyclic points, fourth order determinants and the equation in line coordinates of a conic through the vertices of the triangle of reference.

The fallacy that every angle is a multiple of two right angles is derived by assuming an angle (complex) to satisfy $\tan \theta = i$, and using for any angle A, the formula for $\tan (A + \theta)$. Then $\tan (A + \theta) = \tan \theta$, etc. But the explanation given for the existence of i as a Picard exceptional value of the analytic function $\tan x$ is inadequate

$$\left(\tan x = k, \frac{\sin x}{k} = \frac{\cos x}{1} = \frac{1}{\pm\sqrt{(1 + k^2)}},\right.$$

$$\left. k^2 + 1 = 0 \text{ leads to a denominator 0 and } k = \pm i\right).$$

"Thus no (complex) angle exists whose tangent has either of the values $\pm i$."

The circular points at infinity are introduced to explain the fallacies that the four points of intersection of two conics are collinear, and that concentric circles invert into concentric circles with respect to an arbitrary point. The brief explanation given requires a greater degree of knowledge of projective geometry in the complex plane than the preface of the book would seem to assume.

AUBREY J. KEMPNER
University of Colorado

Elements of Modern Mathematics. By K. O. May. Addison-Wesley, Reading, Mass., 1959. 607 pp. $7.50.

This first-rank book represents a major departure from present freshman level texts, even those of the "modern" school.

The book first introduces some of the ideas and notation of logic and sets, and then proceeds to *use* them for the remainder of its 600 pages. One finds no mere lip service to these topics here; they are put to daily use by the student throughout the course. Other material treated includes plane analytic geometry, a start on differential and integral calculus, probability and statistical inference, and an introduction to abstract algebra.

The strong point of this carefully and well-written book is the author's insistence that things be done right. An especially good feature is the notation for and systematic use of substitution. The text contains some innovations in arrangement. For example, each section contains exercises scattered throughout and problems at the end. Answers to the exercises and some of the problems are given at the end of each section.

The following rather halfhearted criticisms are not intended to detract from the overall excellence of the book. There appear to be the usual number of misprints and minor errors. The section on implicit functions fails to point out that an implicit "function" is often not a function. The axiom systems for fields and Boolean algebras are incomplete since the substitution axioms are omitted (an equivalence relation, rather than the preferable equality, is used in each case). Finally, the treatment of equality of vectors is not in accordance with the otherwise consistent use of equality as identity.

The reviewer feels that *Elements of Modern Mathematics* is an excellent book for the good student and the good instructor, but that it may be somewhat ambitious for the average. Certainly any student who masters a course taught from this book will gain far more mathematical maturity than he would from any other text on this level with which the reviewer is familiar.

W. R. Scott
The University of Kansas

Information Theory and Statistics. By Solomon Kullback. Wiley, New York, 1959. xvii+395 pp. $12.50.

Mathematical information theory is a rather new subject, its main motivations coming from communications engineering. The purpose of this book is to apply the basic concepts of "measuring information" to classical problems of statistics. The motif of the book is that the use of goodness criteria for statistical procedures, based on measures of information, tends to unify an otherwise disorganized subject. My own reaction is that the monumental work of Abraham Wald on statistical decision theory has long provided a basis for unification. The writing is on a fairly advanced technical level. The heart of the book lies in the problems of analysis of variance—the usual univariate linear hypothesis, its somewhat less usual multinomial and Poisson population analogues, and multivariate analysis of variance.

Meyer Dwass
Northwestern University

BRIEF MENTION

High School Mathematics, (Units 1, 2, 3, and 4). By the University of Illinois Committee on School Mathematics. University of Illinois Press, 1959. Teachers' edition about 2,000 pp., $6.00. Students' edition, 618 pp. $2.25.

Many readers will be interested to know that the University of Illinois Committee on School Mathematics, Units 1, 2, 3, and 4, are now available for general sale, in both the student and teacher editions. This is a fascinating example of what can be and is being done to revise and revamp high school mathematics.

On Numerical Approximation. Edited by Rudolph E. Langer. University of Wisconsin Press, 1959. x+462 pp. $4.50.

This volume consists of 21 papers which were delivered at the symposium on numerical approximation conducted by the Mathematics Research Center, United States Army at the University of Wisconsin, Madison, April 21–23, 1958. The contributors are some of the top experts in the field, both from the American continent and abroad. The majority of these papers are mathematical in nature, depending upon functional analysis and linear spaces for basic concepts.

Mathematical Programming and Electrical Networks. By J. B. Dennis. Technology Press and Wiley, New York, 1959. vi+186 pp. $4.50.

This is a research monograph rather than a text. It offers an interesting approach to linear programming by use of analogous electrical networks.

Elementary Statistics with Applications in Medicine and the Biological Sciences. By Frederick E. Croxton. Dover, New York, 1953. vii+376 pp. $1.95.

A reprint of Croxton's 1953 volume.

Probability and Statistical Inference for Engineers. By C. Derman and M. Klein. Oxford University Press, New York, 1959. xii+144 pp. $3.75.

An interesting little book designed to be used in conjunction with several other modern books in presenting a course on probability and statistical inference to undergraduate engineering students.

Statistical Methods in Biology. By Norman T. J. Bailey. Wiley, New York, 1959. ix+200 pp. $4.50.

No mathematics here beyond elementary algebra.

Operations Research—Methods and Problems. By Maurice Sasieni, Arthur Yaspan and Lawrence Friedman. Wiley, New York, 1959. xi+316 pp. $10.25.

Essentially devoted to formulating and solving mathematical models which are based on a no more sophisticated background than that usually provided by differential and integral calculus. A number of interesting and life-like problems are presented which should help project the neophytes into this fascinating new mathematical field.

Strategy and Market Structure. By Martin Shubik. Wiley, New York, 1959. xviii+387 pp. $8.00.

This interesting book suggests that mathematics is currently invading the field of economics through the use of a theory of games designed especially to deal with economic problems. Competition, oligopoly and the general strategic interdependence of economic markets are considered.

The Collected Works of George Abram Miller, Volume V. By George Abram Miller. University of Illinois Press, 1959. xii+474 pp. $7.50.

A fascinating collection of Professor Miller's publications on the subject of finite groups. There are some interesting results and near results contained in these papers, many of which may be unfamiliar to the reader.

Algebra Can Be Fun! By William R. Ransom. J. Weston Walch, Portland, Maine, 1958. ix+195 pp. $2.50.

An interesting collection of high school algebra curiosities.

Amusements in Mathematics and *The Canterbury Puzzles*. By H. E. Dudeney. Dover, New York, 1958. 274 and 266 pp. $1.25 each.

Reprints of Dudeney's 1917 and 1919 books, which still provide interesting puzzles for the nation's puzzlers.

Mathematical Puzzles of Sam Loyd. By Martin Gardner. Dover, New York, 1959. xv+181 pp. $1.00.

This certainly is an all-time puzzle favorite and belongs on every puzzle fan's bookshelf.

Introduction to Symbolic Logic and Its Applications. By Rudolf Carnap. Dover, New York, 1959. xiv+241 pp. $1.85.

An excellent English translation of Carnap's 1954 volume.

Individual Choice Behavior. By R. Duncan Luce. Wiley, New York, 1959. xii+153 pp. $5.95.

This is a research monograph, a probabilistic analysis of choice, by an author who is not afraid of mathematics. Another worthwhile example of modern mathematical inroads into the fields of social science.

On Mathematics and Mathematicians. By Robert Edouard Moritz. Dover, New York, 1958. vii+410 pp. $1.95.

A reprint of Moritz's 1914 *Memorabilia Mathematica*, a collection of anecdotes and quotations concerning famous mathematicians.

Scientific Russian. By George E. Condoyannis. Wiley, New York, 1959. xii+225 pp. $3.50.

An admirable adjunct to the similar volumes on scientific German and scientific French available from the same publisher. Contains a concise description of the structural elements of scientific and technical Russian.

Studies in Mathematical Learning Theory. By R. R. Bush and W. K. Estes. Stanford University Press, Stanford, California, 1959. viii+432 pp. $11.50.

This is a collection of research papers including the most recent developments in the analysis and application of mathematical learning theory. An interesting and worthwhile book of the Stanford Mathematical Studies in the Social Sciences.

Intermediate Algebra, Revised. By J. R. Britton and L. C. Snively. Rinehart, New York, 1959. x+353 pp. $3.00.

A well-known text.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor C. T. Bumer, Clark University, represented the Association at the Academic Convocation of the formal opening of Haberlin Hall at the College of the Holy Cross, on December 8, 1959.

University of Minnesota: Associate Professors W. S. Loud and Bjarni Jonsson have been promoted to Professors; Assistant Professor Steven Orey has been promoted to Associate Professor; Mr. L. R. Harper, Jr. has been promoted to Assistant Professor; Professor M. D. Donsker is spending a sabbatical leave at the University of Aarhus in Denmark.

Dr. Iain Adamson, Queen's University, Belfast, Ireland, has been appointed lecturer at Queen's College, Dundee, Scotland.

Mr. Mansur Arbabi, Seattle University, has accepted a position as Scientist for Booz-Allen Applied Research, Bethesda, Maryland.

Associate Professor Ralph Beatley, Harvard University, has retired with the title Associate Professor of Education, Emeritus and is now teaching half-time in the College of Liberal Arts at Boston University.

Dr. C. R. Cassity, New Mexico Institute of Mining and Technology, has accepted a position as Shock Hydrodynamicist with the General Electric Company, Philadelphia, Pennsylvania.

Dr. K. G. Clemans, USNOTS, China Lake, California, has been appointed Professor at Southern Illinois University, Southwestern Campus, East St. Louis.

Mr. G. R. Costello, Arma Division of American Bosch Arma Corporation, has been promoted to Chief Engineer.

Mrs. Harriet B. Dietz, Auburn High School, Massachusetts, has been appointed Teacher of Mathematics at McArthur High School, Hollywood, Florida.

Mr. E. L. Eagle, Martin Company, Orlando, Florida, has accepted a position as Reliability Manager with the Bendix Aviation Corporation, Ann Arbor, Michigan.

Mr. Herbert Ginsberg, Stanford University, has accepted a position as Statistician with the Fairchild Semiconductor Corporation, Palo Alto, California.

Mr. W. A. Golomski, Oscar Mayer and Company, has been appointed Vice-President of H. J. Mayer and Sons Company, Chicago, Illinois.

Dr. Irwin Guttman, Princeton University, has been appointed Associate Professor at McGill University.

Mr. D. W. Hinkkanen, Boeing Airplane Company, Seattle, Washington, has accepted a position as Research Mathematician with the Chrysler Corporation, Missile Division, Detroit, Michigan.

Mr. D. F. Jordan, Texas Technological College, has been appointed Instructor.

Dr. I. N. Katz, Massachusetts Institute of Technology, has accepted a position as Senior Mathematician with Avco Research and Advanced Development, Wilmington, Massachusetts.

Mr. A. J. Leino, Convair Astronautics Corporation, has accepted a position as Engineer with Lockheed Missile Systems Division, Sunnyvale, California.

Dr. R. D. Mayer, University of Washington, has been appointed Assistant Professor at Idaho State College.

Dr. Fred Meyer, Wayne State University, has accepted a position as Research Scien-

tist with the Boeing Airplane Company, Seattle, Washington.

Dr. J. A. Navarro, General Electric Company, Ithaca, New York, has accepted a position as Statistician with I.B.M. Research, Yorktown Heights, New York.

Mr. C. A. Oster, California Institute of Technology, has accepted a position as Analyst in the Data Processing Division of General Electric Company, Richland, Washington.

Mr. R. L. Page, University of Maine, has been appointed Assistant Professor at Nasson College, Springvale, Maine.

Mr. R. O. Pan, Case Institute of Technology, has accepted a position as Programmer with the Service Bureau Corporation, New York.

Mr. R. R. Parker, General Electric Company, Ithaca, New York, has accepted a position as Senior Engineer with H. R. B. Singer, Inc., State College, Pennsylvania.

Miss Nancy L. Parr, University of Kentucky, has been appointed Instructor at Hanover College, Hanover, Indiana.

Mrs. Jean J. Pedersen, University of Utah, has been appointed Instructor.

Dr. R. P. Peterson, Jr., Matson Navigation Company, has accepted a position as Senior Mathematician with the Burroughs Electrodata Division, Pasadena, California.

Mr. M. B. Richins, Bell Telephone Laboratories, New Jersey, has accepted a position as Senior Operations Research Analyst with the System Development Corporation, Santa Monica, California.

Mr. Paul Roethel, Wisconsin State College, has been appointed Teacher at Berlin High School, Berlin, Wisconsin.

Mr. F. D. Roscoe, Westinghouse Electric Corporation, has accepted a position as Senior Programmer at Lawrence Radiation Laboratories, Livermore, California.

Mr. Sam Rosenberg, Harvard University, has accepted a position as Mathematician with the Texaco Research Laboratory, Bellaire, Texas.

Mr. J. W. Royal, University of Maine, has been appointed Instructor at Merrimack College.

Mr. Yomei Sawanobori, I.B.M. Corporation, New York City, has accepted a position as Senior Mathematician with the International Business Machines Corporation, Washington, D. C.

Associate Professor H. W. E. Schwerdtfeger, McGill University, has been promoted to Professor.

Mr. H. A. Seebald, Sperry Rand Corporation, has accepted a position as Computer Specialist with the System Development Corporation, Santa Monica, California.

Sister Ruthmary, F.S.P.A., Viterbo College, has been appointed Mathematics and Physics Instructor at DePadua High School, Ashland, Wisconsin.

Assistant Professor W. A. Spivey, on leave from the University of Michigan, has been appointed Visiting Associate Professor at the Institute of Basic Mathematics, Harvard Graduate School of Business.

Dr. M. D. Springer, Technical Operations, Inc., Fort Monroe, Virginia, has accepted a position as Systems Operations Analyst with the Allison Division of General Motors Corporation, Indianapolis, Indiana.

Dr. C. E. Stewart, Illinois Institute of Technology, has accepted a position as Research Scientist with the Aeronutronic Division of the Ford Motor Company, Newport Beach, California.

Mr. R. W. Stewart, Nortronics, Anaheim, California, has accepted a position as Electrical Engineer with the Hughes Aircraft Company, Fullerton, California.

Associate Professor W. G. Stokes, Northwestern State College, has been appointed Associate Professor at East Texas State College.

Mr. J. E. Strang, University of Arizona, has accepted a position as Mathematical Programmer at the United States Army Electronic Proving Ground, Fort Huachuca, Arizona.

Associate Professor Takashi Terami, College of St. Thomas, has been promoted to Professor.

Dr. R. J. Warne, University of Tennessee, has been appointed Assistant Professor at Louisiana State University, New Orleans Branch.

Professor A. L. Whiteman, University of Southern California, has been appointed a Member of the Institute for Advanced Study, Princeton, New Jersey, for the year September 1959 to September 1960.

Miss Annie J. Williams, Julian S. Carr High School, has been appointed Supervisor of Mathematics, State Department of Public Instruction, Raleigh, North Carolina.

Dr. C. R. B. Wright, University of Wisconsin, has been appointed Research Fellow at California Institute of Technology.

Professor R. C. Yates, College of William and Mary, has accepted an appointment as Director of the Mathematics Program at the new University of South Florida, Tampa, effective in July, 1960.

Miss Anne Motycka, Proctor High School, Utica, New York, died in July, 1959. She was a member of the Association for six years.

Assistant Professor W. C. Ross, Jr., Knox College, died November 9, 1959. He was a member of the Association for four years.

Professor J. C. Smith, High Point College, High Point, North Carolina, died October 27, 1959. He was a member of the Association for sixteen years.

Professor Emeritus R. D. Stone, Purdue University, died February 27, 1959. He was a member of the Association for forty-one years.

Meeting of Mathematics Division of ASEE

The annual meeting of the Mathematics Division of the American Society for Engineering Education will be held June 20–24 at Purdue University, Lafayette, Indiana. Chairman Sam Selby, University of Akron, has included in the program panel discussions on "The New Role of Undergraduate Mathematics in Engineering" and "The Impact of Digital Computers on Engineering Education" as well as a general session featuring a lecture by Colonel Wernher von Braun.

For further information regarding details of the Annual Meeting write to W. E. Restemeyer, Department of Mathematics, University of Cincinnati.

National Council of Teachers of Mathematics

The Thirty-eighth Annual Meeting of the National Council of Teachers of Mathematics will be held at the Hotel Statler Hilton, Buffalo, New York, April 21–23, 1960. Registration will begin Wednesday afternoon, April 20. The opening general session address "Mathematics and Human Knowledge" will be given by Dr. Carroll V. Newsom, President of New York University, on Thursday evening, April 21.

Additional information may be obtained from Dr. Irvin H. Himmele, NCTM Publicity Chairman, 702 City Hall, Buffalo 2, N. Y. or Louis F. Scholl, Chairman of Local Arrangements, 728 City Hall, Buffalo 2, N. Y.

AMS Summer Institute on Finite Groups

With the support of the National Science Foundation, a Summer Institute on Finite Groups will be held at the California Institute of Technology from August 1 to August 28, 1960. Two distinguished group theorists have been invited from abroad and have accepted the invitation. These are Professor Graham Higman of Oxford University and Professor Helmut Wielandt of the University of Tübingen. Participation in the Summer Institute will be by invitation, but the lectures and seminars will be open to interested mathematicians who happen to be in the area. The Program Committee consists of Professors Richard Brauer, Richard Bruck, H. S. M. Coxeter, Robert Dilworth, Herbert Ryser, and Marshall Hall, Jr., Chairman.

Southern Regional Graduate Summer Session in Statistics

The 1960 Southern Regional Graduate Summer Session in Statistics will be held at the University of Florida at Gainesville from June 20 to July 29, 1960. The University of Florida, North Carolina State College, Virginia Polytechnic Institute and Oklahoma State University have agreed to operate a continuing program of graduate summer sessions in statistics to be held at each institution in rotation. The first such session was held at Virginia Polytechnic Institute in the summer of 1954.

It is the purpose of this program to serve: (1) teachers of introductory statistical courses and college teachers of mathematics who want formal training in modern statistics; (2) research and professional workers who want intensive instruction in basic statistical concepts and modern statistical methodology; (3) professional statisticians who wish to keep informed about advanced specialized theory and methods; (4) prospective candidates for graduate degrees in statistics; and (5) graduate students in other fields who desire supporting work in statistics.

The session will last six weeks and courses will carry three semester hours of credit. Not more than two courses may be taken for credit at any one session. The summer work in statistics may be applied as residence credit at any one of the cooperating institutions, as well as certain other universities, in partial fulfillment of the requirements for a graduate degree. The program may be entered at any session, and consecutive courses will follow in successive summers.

The courses to be offered in statistics in 1960 at the University of Florida are as follows: Statistical Methods I, II and III, through Sample Survey Methods; Statistical Theory I, II and III, including Probability, Inference and Least Squares; Statistical Problems; Advanced Statistical Inference; and Response Surfaces.

Requests for application blanks should be addressed to Dr. Herbert A. Meyer, Statistical Laboratory, University of Florida, Box 3568, Gainesville, Florida.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. L. Alder, Secretary, announces that the following 150 persons have been elected to membership by the Board of Governors on applications duly certified.

VIJA ABELE, B.A. (Ohio Northern) Asst. Research Physicist, Mound Laboratory

ROCHELLE ABEND, M.A. (Illinois) Instr., Kansas State University

PFC. ALFRED ADAMSKI, B.A. (Ripon Coll.) Math. Asst., United States Army

JOHN E. ALLEN, B.S. (Louisiana Polytech. Inst.) Grad. Asst., Oklahoma State University

LEE W. ANDERSON, Ph.D. (Tulane) Asst. Prof., University of Oregon

RICHARD J. ANDRES, Student, College of the Holy Cross

HUBERT AUSTIN, M.A. (Ball S.T.C.) Asst. Prof., Tri State College

ABDUL K. AZIZ, Ph.D. (Maryland) Asst. Prof., Georgetown University

RICHARD F. BAKER, Student, University of Oklahoma

MARGARET G. BANTA, A.M. (Columbia) Chairman of Dept., Butler High School, New Jersey

JOHN F. BARTH, B.A. (Willamette) Grad. Asst., Michigan State University

JOHN H. BIGGS, A.B. (West Virginia) Grad. Asst., West Virginia University

GORDON D. BJONERUD, B.S. (Northern S.T.C.) Teacher, High School, Bound Brook, New Jersey

WYLAND L. BLANCHARD, Student, Amherst College

RAYMOND E. BOCHE, B.S. (Calif. State Polytech.) Programmer, Lockheed Missiles & Space Division

RONALD L. BOHUSLOV, M.A. (California, Berkeley) Instr., Oakland City College

THOMAS B. BOONE, JR., M.A. (Northwest S.C. of Louisiana) Instr., Northwestern State College

BROTHER ROBERT N. COSTA, B.S. (St. Mary's Coll., Calif.) Chairman of Dept., Chaminade High School, Los Angeles, California

STEVEN J. BRYANT, Ph.D. (Missouri) Asst. Prof., Fresno State College

JACK W. CARLYLE, M.S. (Washington) Asso., Electrical Engineering, University of California

DAVID C. CARRAD, Student, Montclair Academy, New Jersey

WARREN P. CASH, JR., B.S. (Texas) Grad. Student, University of Texas

DAVID J. CAVENY, B.A. (Western Carolina Coll.) Grad. Asst., University of Kentucky

SHU-KWAN CHAN, Student, Massachusetts Institute of Technology

HERBERT H. CHU, M.S. (Oklahoma S.U.) Instr., American University

HENRY C. CLARKE, M.S. (California) Teacher, Beverly Hills High School, California

WILLIAM F. CLARY, M.D. (Tennessee) Psychiatrist, Springfield, Missouri

RAY W. CLEVELAND, M.A. (Columbia T.C.) Teacher, Rahway High School, New Jersey

ROBERT L. COOLEY, M.S. (Purdue) Part-time Instr., Purdue University

EDWARD J. CREAGER, JR., Student, Xavier University

DOUGLAS A. DAVIDS, M.S. (Newark Coll. Engineering) Engineer, Applied Physics Lab., Johns Hopkins University

WILLIAM E. DEMALIGNON, M.A. (Wisconsin) Asst. Prof., State University of South Dakota

ROBERT J. DIAMOND, Ph.D. (Calif. Inst. Tech.) Asst. Prof., Los Angeles State College

ALPHONSO J. DIPIETRO, Ph.D. (George Peabody Coll.) Asst. Prof., Eastern Illinois University

RALPH L. DISNEY, M.S. (Johns Hopkins) Asso. Prof., Industrial Engr., University of Buffalo

HENRY C. DIXON, JR., A.B. (Bowdoin) Grad. Student, Boston College

JAMES J. DODD, M.S. (Illinois) Instr., Whitewater State College

EDWIN D. ECKER, M.S. (Illinois) Instr., MacMurray College

MAGDI EL-KAMMASH, M.P.H. (North Carolina) Research Asst., Duke University

ARNOLD ESTEP, Student Asst., University of Kentucky

JANE L. EVANS, M.A. (Alabama) Instr., St. Petersburg Junior College

HAROLD E. FASSBERG, M.A. (Pittsburg) Deputy Director, Corp. for Economic & Industrial Research

WYMAN G. FAIR, M.A. (Missouri) Instr., University of Missouri

WILLIAM M. FITZGERALD, M.S. (Michigan) Teaching Fellow, University High School, Ann Arbor, Michigan

RICHARD A. FLECK, B.S. (Eastern Kentucky S.C.) Teaching Fellow, University of Kentucky

CHARLES H. FRANKE, M.A. (Yale) Asst. Instr., Rutgers University, Newark

MRS. MARCELLE FRIEDMAN, Student, Seton Hall University

CLAYTON H. GARDNER, M.S. (Illinois) Chairman of Dept., White Plains High School, New York

MARTIN GARDNER, B.A. (Chicago) Editor, Scientific American

SEYMOUR GEISSER, Ph.D. (North Carolina) Mathematician, National Institute of Mental Health

ALLAN GEWIRTZ, B.S. (Brooklyn) Lecturer, Brooklyn College; Engineer, Waldorf Indus. Controls

LARRY K. GILCHRIST, B.A. (Knox Coll.) Grad. Asst., Oklahoma State University

RICHARD P. GOBLIRSCH, Ph.D. (Wisconsin) Asst. Prof., University of Rochester

JOHN K. GOODE, B.S. (Texas A & M) Grad. Student, Vanderbilt University

LARRY C. GROVE, Student, University of Minnesota

DAVID G. HALMSTAD, A.B. (Harvard) Grad. Student, University of Wisconsin

JOHN G. HARRINGTON, B.S. (Massachusetts) Asst., University of Illinois

VERN D. HIEBERT, M.S. (Illinois) Asst. Prof., Oregon College of Education

JOSEPH B. HOFFERT, M.A. (Wyoming) Asst. Prof., Drake University

MARTIN O. HOLOIEN, B.S. (Moorhead S.C.) Instr., North Dakota Agricultural College

EDWIN N. HUMPHREY, A.B. (Oklahoma) President, Chickasha Milling Company, Oklahoma

ROBERT J. HUSKEY, Student, University of Oklahoma

MICHAEL A. IWANOWICZ, Student, Bowdoin College

HARVEY JACKINS, President, Personal Counselors Inc., Seattle, Washington

MARC Q. JACOBS, Student, University of Oklahoma

S. E. J. JOHNSEN, M.S. (Purdue) Instr., Purdue University

BJARNI JONSSON, Ph.D. (California) Prof., University of Minnesota

MRS. MARJORIE W. KATZ, Student, University of Chicago

SAROOP K. KAUL, Ph.D. (Delhi U.) Instr., University of Rochester

CARL D. KEEFER, B.S. (Millersville S.T.C.) Grad. Student, Bucknell University

HUBERT C. KENNEDY, M.A. (Michigan) Grad. Fellow, St. Louis University

PAUL KROLL, A.M. (Columbia) Senior Manager, New York State Division of Employment, Brooklyn

DONALD E. KNUTH, Student, Case Institute of Technology

HERBERT C. KUGEL, Student, Los Angeles State College

H. ELTON LACEY, B.A. (Abilene Christian Coll.) Grad. Asst., Abilene Christian College

VINCENT J. LAMPHIER, B.A. (Manhattan Coll.) Math. Programmer, I.B.M., Kingston, New York

WALKER H. LAND, JR., B.S. (Virginia Polytech. Inst.) Electrical Engineer, Melpar

STEPHEN LANGE, A.B. (Connecticut) Teacher, Storm King School, Cornwall-on-Hudson, New York

EDWARD H. LARSON, M.S. (M.I.T. Senior Member Technical Staff Radio Corporation of America

LOREN C. LARSON, B.S. (Bethany Coll.) Grad. Asst., University of Kansas

DAVID M. LEDERMAN, Student, University of California, Berkeley

LEON LEVINE, M.S.E.E. (M.I.T.) Group Head, Hughes Aircraft Co.

MARK LEVINSON, M.S. (Polytech. Inst., Brooklyn) Asst. Prof., Mech. Engr., Oregon State College

THOMAS S. LOGSDON, B.S. (Eastern Kentucky S.C.) Grad. Student, University of Kentucky

FRANK E. MCFARLIN, M.S. (Oklahoma) Project Engineer, I.B.M., Endicott, New York

PHILLIP S. MITCHELL, A.A. (Valley Junior Coll.) Student, University of California, Riverside

MAJOR REX K. MOORHEAD, M.A. (Bowling Green State U.) Instr., United States Air Force Academy

WALLACE J. MORRISON, B.A. (Queen's U.) Head of Dept., Danforth Technical School, Toronto

WALTER P. MORSE, Ph.D. (Florida) Asso. Prof., University of Florida

MARGARET M. MORTON, M.A. (Michigan) Teacher, North Platte Senior High, Nebraska

FRANK MULLAN, B.S. (Dayton) Teacher. Chaminade High School, Mineola, New York

BRUCE M. OWEN, Student, Millbury Memorial High School, Massachusetts

KENNITH R. OWENS, B.A. (Baylor) Teacher, St. Mark's School of Dallas, Texas

CAPT. JAMES T. PACE, Chief, Air Defense Systems Division, Tinker Air Force Base

ROBERT E. PARENT, M.Ed. (Miami) Teacher, Coral Gables Senior High School, Florida

ARTEMAS M. PICKARD, A.B. (Brown) Mathematician, David W. Taylor Model Basin

BENJAMIN F. PLYBON, M.S. (Carnegie Inst. of Tech.) Instr., Marshall College

EDWARD L. PUGH, B.E.E. (Santa Clara) Asso., Engineer, Ryan Aeronautical Company

STANLEY R. PURVEAR, Student, Monmouth College

DORIS M. QUINN, M.A. (N.Y.S. Coll. for Teachers) Teacher, Bethlehem Central Senior High School, Delmar, New York

ALBERT L. RABENSTEIN, Ph.D. (M.I.T.) Asst. Prof., Allegheny College

JAMES M. REID, B.S. (Hofstra Coll.) Teacher, Walt Whitman High School, New York City

STANLEY F. ROBINSON, B.A.S. (California, Los Angeles) Teacher, Pasadena City College

JORGE E. RODRIGUEZ-SANABIA, B.S. (Puerto Rico) Instr., University of Puerto Rico

DAVID A. ROSE, M.A. (California) Senior Dynamics Engineer, Convair

GLENN M. ROE, Ph.D. (Minnesota) Physicist, General Electric Research Laboratory

MICHAL RUKIEWICZ, M.S. (Poland) Lecturer, Univ. of Ottawa, Carleton Univ. & St. Patrick Coll., Ottawa, Ontario

SAM SCHAUERMAN, JR., M.A. (Colorado S.C.) Inst., Otero Junior College

DONALD R. SCHUETTE, M.A. (Michigan) Asst. Prof., Drake University

JACOB SCHWARTZ, B.S. (New York) Certified Public Accountant, Goldfinger & Schwartz, New York
CHARLES W. SEEKINS, Ph.D. (Calif. Inst. of Tech.) Prof., Occidental College
MRS. MAUDDEAN T. SEWARD, B.S. (Tenn. A & I S.C.) Teacher, Memphis City Board of Education
RICHARD T. SHANNON, Student, Fordham University
RICHARD E. SHERMOEN, M.S. (North Dakota Agric. Coll.) Instr., North Dakota Agricultural College
REV. EDWIN J. SHERRY, M.A. (Fordham) Grad. Student, Fordham University
IONEL S. SHOLA, Student, University of Rhode Island
JOSEPH D. H. SIDLEY, A.B. (Georgetown U.) Grad. Asst., Georgetown University
SISTER LOYOLA CULHANE, M.A. (Indiana) Teacher, Providence High School, Chicago, Illinois
SISTER M. JANE FRANCES, M.A. (Catholic U.) Asso. Prof., Chestnut Hill College
SISTER MARTHA FRANCIS, B.S. (Annhurst Coll.) Teacher, St. Anne's Academy, Swanton, Vermont
MELVIN T. SILLITO, M.Ed. (Alberta) Instr., Lethbridge Junior College
LEHI T. SMITH, Ed. D. (Stanford) Asst. Prof., Arizona State University
SANDRA N. SMITH, B.S. (Howard) Teacher, MacFarland Junior High School, Washington, D.C.
JOHN F. SOKOL, M.A. (Amer. International Coll.) Instr., Ward School of Electronics, Hartford, Connecticut
NORMAN D. SOSSONG, Student, Walla Walla College
JEROME L. STANDIG, M.S. (C.C.N.Y.) Mathematician, Fort Monmouth, New Jersey
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ITINERARIES OF VISITING LECTURERS, 1960

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Mar. 28–29	University of Alberta, Edmonton, Alta. (E. S. Keeping)
Mar. 31–Apr. 1	University of British Columbia, Vancouver, B. C. (R. D. James)
Apr. 4–5	Western Washington College of Education, Bellingham, Wash. (Harvey M. Gelder)
Apr. 6	St. Martin's College, Olympia, Wash. (Rev. John Raymond, O.S.B.)
Apr. 7–8	Seattle University, Seattle 22, Wash. (T. S. Chihara)
Apr. 11–13	Washington State University, Pullman, Wash. (S. G. Hacker) and University of Idaho, Moscow, Ida. (K. A. Bush)
Apr. 15–19	Whitman College, Walla Walla, Wash. (W. L. Hutchings) and Walla Walla College, College Place, Wash. (L. G. Schlotthauer)
Apr. 21–23	College of Idaho, Caldwell, Ida. (Boyd Henry)
May 2–3	Eastern New Mexico University, Portales, N. Mex. (E. A. Propes)

May 5–6	New Mexico State University, University Park, N. Mex. (Earl Walden)
May 9–10	California State Polytechnic College, San Luis Obispo, Calif. (M. E. Whitson)
May 12–13	Fresno State College, Fresno 26, Calif. (G. Don Alkire)
May 16–17	Humboldt State College, Arcata, Calif. (H. D MacGinitie)
May 19–20	Chico State College, Chico, Calif. (Lloyd M. Cook)
May 23–24	Sacramento State College, Sacramento, Calif. (Stanley P. Hughart)
May 25	University of Utah, Salt Lake City, Utah (C. R. Wylie, Jr.)
May 26–27	Brigham Young University, Provo, Utah (H. J. Fletcher)

THE OCTOBER MEETING OF THE IOWA SECTION

Members of the Iowa Section of the Mathematical Association of America met jointly with Teachers of Mathematics at the 29th Annual Conference of Teachers of Mathematics at the State University of Iowa, October 16, 1959. The morning meeting included an address by Lenore John, University of Chicago, who spoke on "The Recommendations of the School Mathematics Study Group for Grades 7 and 8." Mr. Henry Swain of the New Trier Township High School, Winnetka, Illinois also gave an address entitled "The Recommendations of the School Mathematics Study Group for Grade 9."

At the afternoon meeting, the first topic, "The Minnesota National Laboratory for the Improvement of Secondary Mathematics," was presented by Frank Wolf, Carleton College. A general discussion period followed this address.

E. L. CANFIELD, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual fall meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Delaware on Saturday, November 28, 1959, with Professor Marguerite Lehr, Bryn Mawr College, presiding. There were 54 present at the meeting, including 44 members of the Association.

At the Business Meeting the following officers were elected: Chairman, Professor W. S. Lawton, Temple University; Third member of the Executive Committee, Professor R. F. Jackson, University of Delaware; Professor John Brown, University of Delaware, was appointed chairman of a committee to act as Liaison agent to State Authority.

Professor Lawton reported on the progress of the High School contest conducted in the Philadelphia area. Professor Lehr reported on the Professional Standards Conference.

The following program was presented:

1. *Probabilistic models in psychology for the study of choice behavior*, by Professor R. D. Luce, University of Pennsylvania.

An organism making choices from a finite set of alternatives can, sometimes, be characterized as a probability distribution over the alternatives. The heart of the psychological problem, then, is to ascertain the laws that relate these distributions to one another, both in a static and dynamic sense. One general static law, several specific static laws, and one general dynamic law are described, and their application to problems in psychophysics, gambling behavior, and learning are sketched. A number of unresolved mathematical problems are outlined that arise in attempting to test these results empirically.

2. *On some extremal problems in geometry*, by Professor A. S. Besicovitch, University of Pennsylvania. (By invitation.)

Three extremal problems are considered: (1) The classical isoperimetric problem about the maximum area of a domain with fixed length of the boundary. Steiner's solution is given and analyzed. (2) The problem on the maximum area of a set of a given diameter. An elegant solution of W. Feller is given. (3) The problem on the minimum area of a set of constant width is treated by expressing the area bounded by an arc and the tangents at the end points in terms of curvature.

3. *College mathematics for the prospective graduate student*, by Professor Bernard Epstein, University of Pennsylvania.

Many beginning graduate students are handicapped by the lack of understanding of the fundamental ideas of real analysis. In too many cases "advanced calculus," as a mere continuation of "elementary calculus," is not meeting this need. A "little real-variable" course is needed containing, in particular, material dealing with the topology of the reals, including the Bolzano-Weirstrass and Heine-Borel Theorems. A course dealing with some phases of higher algebra is also highly desirable. These courses could replace, for the prospective graduate student, traditional courses which are largely devoid of mathematical content—in particular, the usual manipulative courses in differential equations and statistics.

4. *Work of the School Mathematics Study Group at Boulder and Ann Arbor*, by Professor V. H. Haag, Franklin and Marshall College.

The School Mathematics Study Group at Boulder and Ann Arbor in the summer of 1959, consisting of more than eighty mathematicians and teachers, not only prepared preliminary editions of sample texts for grades seven through twelve, but found themselves learning a great deal about each other's problems. It is believed that this liaison between the two groups, the mathematicians and the teachers, was a valuable contribution to the solution of problems of mathematics education.

5. *Liaison problems in collegiate mathematics today—with the high school*, by Mr. M. A. Linton, Jr., William Penn Charter School.

Liaison between high school and college teachers of mathematics appears to be stronger today than at any previous time. Most, if not all of the agencies now at work in reshaping the elementary and secondary school mathematics curricula depend upon the cooperative efforts of members of both groups. As a result, not only are the new courses sound in mathematical content, but they are also teachable. An important by-product of this cooperation is increased awareness by both college and school teachers of the problems of each other. Ultimately, the reforms being wrought in school mathematics courses will affect the college curriculum as well.

F. L. DENNIS, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-D.C.-VIRGINIA SECTION

The winter meeting of the Maryland-D.C.-Virginia Section of the Mathematical Association of America was held on December 5, 1959, at American University, Washington, D. C. Professor E. E. Floyd, of the University of Virginia, presided. The total attendance was 151 including 107 members.

At the business meeting, following the program, the registration fee for meetings was increased from twenty-five cents to fifty cents to cover rising costs of postage and occasional expense for outside invited speakers. Mr. W. H. Norris, of Norfolk, described the progress made in improving the high school contest program.

The following papers were presented:

1. *New contributions to educational research from operations research*, by Professor J. M. Long, College of William and Mary.

The paper discussed four points: (1) the nature and development of operations research; (2) several aspects of the educational system which might be susceptible to operations research methods; (3) queueing theory was briefly described and analyzed in relation to the educational process; (4) the use for education of methods associated with the optimum solution problem was considered.

2. *On the proof of Gödel's incompleteness theorem*, by Professor D. B. Lloyd, D. C. Teachers College.

In 1931 Kurt Gödel proved an outstanding theorem to the effect that any broad mathematical system, such as arithmetic, is an incomplete theory in the sense that there are theorems therein which are true but which cannot be proved within the framework of its own axiomatic structure

Gödel mapped the metamathematical logic into an arithmetical model, converting the proof into a number-theoretic one. The paper sketched the techniques of this unique proof and the resulting impacts upon the modern understanding of the inherent nature of mathematics.

3. *Elementary combinatorial problems related to a mail-sorting device*, by Dr. A. J. Goldman, National Bureau of Standards.

Mail to any of r destinations enters the device k letters at a time. After each k letters enter, the device "asks" to which destination it contains most letters, and drops out all letters to this destination. It is shown that a capacity of $(r-1)(k-1)+k$ letters (but no fewer) suffices to prevent overflow, and that any capacity in excess of rk letters will be wasted; for realistic values (*e.g.*, $r=200$, $k=3$) these numbers are much lower than the upper bound of r^2k obtained by another investigator. These results were obtained jointly with Mrs. B. K. Bender, as part of a project supported by the Post Office Department, Office of Research and Engineering.

4. *A note on polynomial interpretation*, by Dr. C. H. Frick, Naval Weapons Laboratory, Dahlgren, Virginia.

By assigning errors at the data points, Chebychev polynomials were used without the inconvenience of normalizing arguments or establishing data points with irrational arguments.

5. *The calculus of relations*, by Dean W. L. Duren, University of Virginia.

This was an elementary treatment of the calculus of relations of the form $F(x, y)=C$. Without regarding the relation as an implicit function it leads to the tangent relation $F_x dx+F_y dy=0$, where dx and dy are local coordinates with respect to axes centered at (x, y). Conversely, the integral calculus gives integrals of differential equations of the form $M(x, y)dx+N(x,y)dy=0$. The uniqueness of the integral is proved but only in the case of separable variables is the existence of the integral an elementary problem. It is pointed out that the uniqueness problem for integrals is essentially different from that for solutions.

6. *Some examples of completely semiassociative and noncommutative loops and algebras*, by Professor Volodymyr Bohun-Chudyniv, Morgan State College.

An algebra we call *completely semiassociative* if the number α of nonassociative cases in 6 products in all ordered triplets u_α, u_β, u_γ, of the basal units, not belonging either to the center or to the same subalgebra, equal the number β of *associative cases*. The aims of this paper are to determine some examples of completely semiassociative algebras and loops and consider some of their properties.

7. *Hypercomplex methods in the theory of partial differential equations*, by Professor J. M. Horvath, University of Maryland.

Given a homogeneous partial differential operator with constant coefficients D, an infinite dimensional algebra Q can be associated with it, which permits to construct a complete set of polynomials Y, solutions of $DY=0$ (Proc. Amer. Math. Soc., vol. 9, 1958, pp. 569–575). If D is elliptic, then every solution of $Df=0$ can be approximated by linear combinations of the Y. If the number of variables is two, a finite-dimensional quotient algebra A of Q and D-analytic of functions with values in A can be defined. The theory of the D-analytic functions yields further results concerning the approximation of solutions of $Df=0$.

8. *An algorithm for the determination of divisibility of one natural number by another*, by Professors Herta T. and A. H. Freitag, Hollins College, Virginia.

A new criterion for determining divisibility of one natural number by another is developed. In case of divisibility, this algorithm constructs the quotient without the necessity of estimating trial divisors. The operations in this procedure are less involved than those in ordinary division. Certain number properties become apparent.

9. *Curvature of* $r^n=\cos n\theta$, by Professor R. C. Yates, College of William and Mary.

(See pp. 275–278 of this issue.)

10. *What is the mathematical theory of knots?* by Professor R. H. Fox, Princeton University. (Invited address.)

Knot theory deals with the topological classification of the ways in which a simple closed curve can be imbedded in 3-dimensional space. To each such imbedding is associated the fundamental group of the complementary space, and thereby an integral polynomial $\Delta(t)$. Thus to the trivial knot is associated $\Delta(t)=1$, to the overhand knot $\Delta(t)=1-t+t^2$, to the figure-of-eight $\Delta(t)=1-3t+t^2$, to the square knot or to the granny knot $\Delta(t)=(1-t+t^2)^2$. There are many other invariants, some of greater power and subtlety. With these it is possible to distinguish, for example, between the square knot and the granny. However, a complete description of all the imbedding types is very far from being realized.

D. B. LLOYD, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 30, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK, City College, New York, April 2, 1960.

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Norman, April 15–16, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

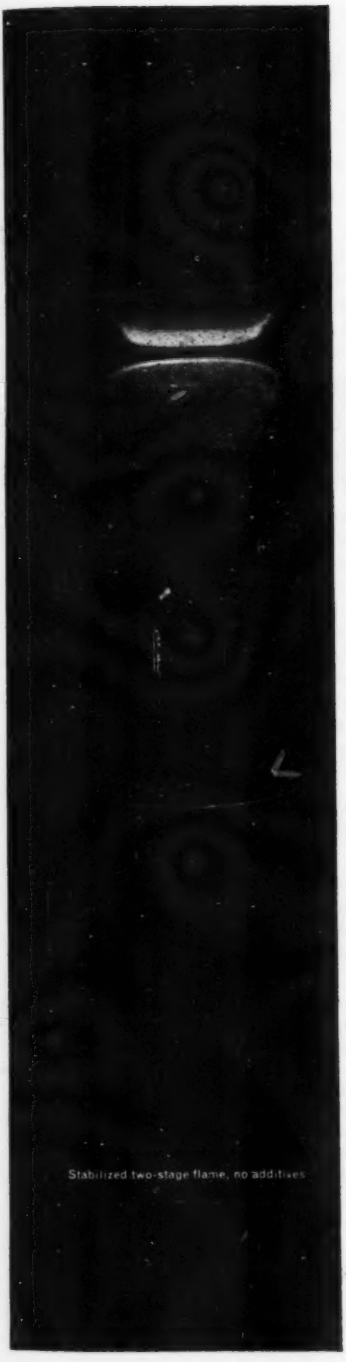
SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April 8–9, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.



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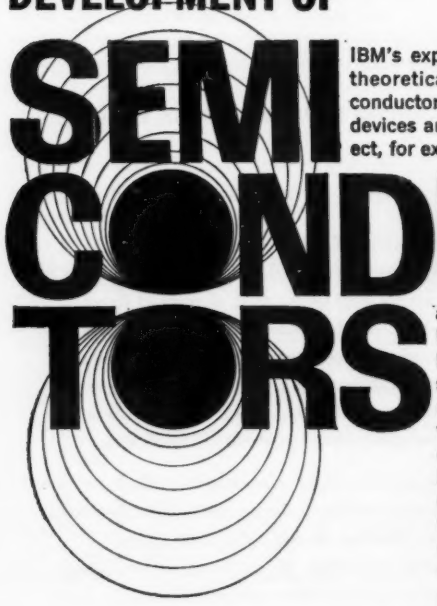

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